14. Consider two-dimensional steady state conduction in a region, 2 cm × 2 cm, with the boundary conditions as shown in Fig. Problem 8.14. For the material, k = 60 W/ (mC) and there is internal heat generation at a rate of 10^7 W/m^3. Using finite difference method, calculate the unknown node temperatures.

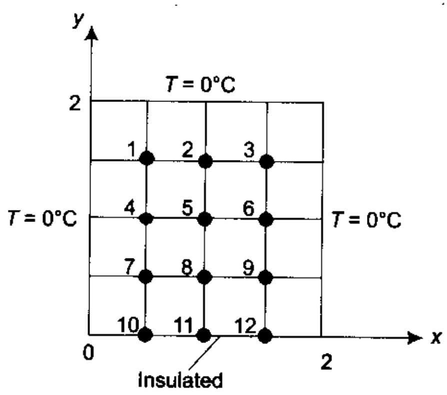

FIGURE Problem 8.14 Two-dimensional steady state conduction

One-dimensional transient conduction:

15. A very thick copper plate (k = 386 W/(mC), aa = 11×10^{-5} m^2/s) is initially at 400°C. Suddenly, its surface temperature is lowered to 20°C. Considering the plate as semi-infinite plate and using a mesh size Δx = 1 cm, calculate the temperature at x = 5 cm from the surface, 2 min. after lowering the surface temperature.
16. A water main is buried below the surface of soil which is initially at an uniform temperature of 25°C. Suddenly, the surface temperature drops to –30°C and is maintained so for a period of 60 days. Determine the depth at which the water mains must be placed to avoid freezing of water. Take properties of soil as: ρ = 2050 kg/m^3, k = 0.52 W/(mC), C_p = 1840 J/(kgK), α = 0.138×10^{-6} m^2/s. (Hint: Consider the soil as semi-infinite medium; calculate temperatures at distances upto 6 m below the surface and find the depth at which the temperature would be 0°C, by interpolation).
17. A 6 cm thick steel plate ($\alpha = 1.6 \times 10^{-5}$ m^2/s, k = 60 W/(mC)), is initially at an uniform temperature of 250°C. It is suddenly exposed to a cold air stream at 20°C on both the surfaces, with a heat transfer coefficient of 350 W/(m^{2}C). Determine the centre plane temperature at τ = 5, 10 and 15 min. from starting of cooling. Use explicit formulation with a mesh size of Δx = 1 cm.
18. Two ends of a steel rod 1.2 cm diameter and 2.5 m long, are maintained at 250°C and 50°C and the curved surface of the rod is perfectly insulated. Suddenly, an electric current is passed through the rod, causing heat generation in the rod at an uniform rate of 3000 W/m^3. Find the temperature distribution in the rod for the first five time increments. Take k = 35 W/(mC) and $\alpha = 1.5 \times 10^{-5}$ m^2/s.

Two-dimensional transient conduction:

19. The L-bar shown in Fig. Problem 8.9 is initially at an uniform temperature of 200°C. Its top surface is suddenly exposed to convection with an air stream at 20°C with a convection coefficient of 80 W/(m^{2}C). Bottom surface is maintained at 200°C throughout and the left and right surfaces are insulated as shown. Taking k = 15 W/(mC) and $\alpha = 3.2 \times 10^{-6}$ m^2/s, calculate the temperature of node 3 after 1, 3, 5, 10 and 30 min. Use explicit formulation.
20. A steel bar of 3 cm × 3 cm cross-section is initially at an uniform temperature of 500°C. ($\alpha = 1.0 \times 10^{-5}$ m^2/s, k = 35 W/(mC)). Suddenly, all the 4 surfaces of the bar are exposed to an air stream at 20°C with a heat transfer coefficient of 120 W/(m^{2}C). Using explicit formulation and a mesh size of $\Delta x = \Delta y$ = 0.5cm, calculate the centre temperature at τ = 1, 5 and 10 min. after the start of cooling. (Hint: Use symmetry consideration—consider only a quarter of the cross-section).

CHAPTER

9

Forced Convection

9.1 Introduction

In the previous chapters, we studied about conduction heat transfer, where heat transfer was a molecular phenomenon and was considered mainly in solids; convection was mentioned only in passing and was considered only as a boundary condition while analysing conduction heat transfer.

In convection heat transfer, there is a flow of fluid associated with heat transfer and the energy transfer is mainly due to bulk motion of the fluid. When the flow of fluid is caused by an external agency such as a fan or pump or due to atmospheric disturbances, the resulting heat transfer is known as 'Forced convection heat transfer'; when the flow of fluid is due to density differences caused by temperature differences, the heat transfer is said to be by 'Natural (or free) convection'. For example, if air is blown on a hot plate by a blower, heat transfer occurs by forced convection, whereas, a hot plate simply hung in air will lose heat by natural convection.

In this chapter, we shall study about forced convection heat transfer. Since there is a flow of fluid involved in convection heat transfer, it is clear that the flow field will influence the heat transfer greatly. Mathematical solution of convection heat transfer will, therefore, require the simultaneous solution of differential equations resulting by the application of conservation of mass, conservation of momentum and conservation of energy, under the constraints of given boundary conditions. For a three-dimensional fluid flow, mathematical solution of the resulting differential equations is a formidable task and it is usual to make many simplifying assumptions to get a mathematical solution. Still, it must be stated that exact mathematical solutions, even for simple convection heat transfer cases, are rather complicated and it is common practice to resort to empirical relations for solutions of problems involving convection heat transfer. These empirical relations are obtained by researchers after performing large number of experiments for several practically important situations and are presented in terms of non-dimensional numbers.

In this chapter, we shall first describe the physical mechanism of forced convection and then mention about the convective heat transfer coefficient and various factors affecting the same. Then, we shall discuss about velocity and thermal boundary layers. Application of conservation of mass, momentum and energy in respect of the boundary layer will be demonstrated next. We shall not rigorously solve these equations, but will only mention the methods of solution, since our emphasis will be on practical solutions with the use of empirical relations. Then, we present several empirical relations to determine friction and heat transfer coefficients for flow over different geometries such as a flat plate, cylinder and sphere for flow under laminar and turbulent conditions. Finally, flow inside tubes will be considered and determination of heat transfer coefficient by analogy with the mechanism of fluid flow will be explained.

9.2 Physical Mechanism of Forced Convection

Consider a hot iron block whose surface is at a temperature T_s. Let this surface be cooled by a fluid at a temperature T_a, flowing over its surface at a velocity U, as shown in Fig. 9.1.

We know that heat will be carried away from the hot iron block by the flowing fluid and the block will cool. We also know that if the velocity of the fluid is increased, more heat is carried away and the block will be cooled faster. For the purpose of analysis, we quantify the preceding statement by a dimensionless number called,

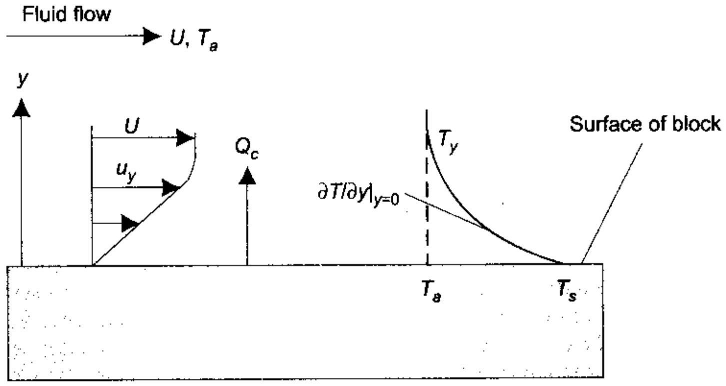

FIGURE 9.1 Temperature and velocity distribution in laminar, forced convection over a hot block

'Reynolds Number', in honour of Osborne Reynolds, an English scientist. Reynolds number is defined as follows:

$$Re = U \frac{x \cdot \rho}{\mu} \quad \text{...(9.1)}$$

where U = mean velocity of flow, m/s

ρ = densiy of fluid, kg/m^3

μ = dynamic viscosity of fluid, kg/(ms), and

x = a characteristic dimension of the flow passage, equal to the linear distance along the flow direction in the case of a flat plate or the pipe diameter in the case of a flow through a pipe. For non-circular passages (such as a square or rectangular passage), the characteristic dimension in Eq. 9.1 is the 'equivalent diameter', defined as:

$$d_e = \frac{4 \cdot A_c}{P} \quad \text{...(9.2)}$$

where d_e = equivalent diameter, m

A_c = area of cross-section, m^2, and

P = wetted perimeter, m

For a rectangular cross-section of breadth 'a' and height 'b', we get from Eq. 9.2:

$$d_e = \frac{4 \cdot A_c}{P} = \frac{4 \cdot a \cdot b}{2 \cdot (a+b)} = \frac{2 \cdot a \cdot b}{(a+b)} \quad \text{...(9.2,a)}$$

And, for an annulus formed by a tube of outer diameter d_1 placed within a tube of inner diameter d_2, equivalent diameter is calculated as:

$$d_e = 4 \cdot \frac{\frac{\pi}{4} \cdot (d_2^2 - d_1^2)}{\pi \cdot (d_1 + d_2)} = d_2 - d_1 \quad \text{...(9.2,b)}$$

Note that Eq. 9.2 is used in connection with the calculation of pressure drop for flow through an annulus; but, for the case of heat transfer, say from a hot fluid flowing through the inner tube to a cold fluid flowing through the outer tube, since the heat transfer occurs only through the surface of the inner tube, we use for the equivalent diameter:

$$d_e = 4 \cdot \frac{\frac{\pi}{4} \cdot (d_2^2 - d_1^2)}{\pi \cdot d_1} = \frac{d_2^2 - d_1^2}{d_1} \quad \text{...(9.2c)}$$

If the Reynolds number is below a certain value, as determined by experiments, the flow is laminar; i.e. the fluid layers move parallel to each other in an orderly manner. As the velocity of flow increases, i.e. as the value of Reynolds number increases, there is more disorder in the fluid and the fluid flow becomes 'turbulent'; fluid 'chunks' move at random and obviously the heat transfer increases, since these chunks of fluid carry the heat with them. Transition from laminar to turbulent flow occurs not at a fixed value of Reynolds number, but occurs in a range called 'transition range of Reynolds numbers'. For example, for flow through a pipe, at values of Reynolds number below 2300 the flow is laminar, for values above 4000 the flow is turbulent and in between is the transition range. Value of Reynolds number is affected by fluid properties, dimension of flow passage and also by surface conditions.

Fig. 9.1 also shows the velocity profile and the temperature profile for laminar flow. The velocity profile is parabolic. As the flowing fluid comes in contact with the surface of the block, a thin layer adheres to the surface and essentially remains stationary with zero velocity; this phenomenon is known as 'no slip condition' in the terminology of fluid mechanics. The fluid layer adjacent to this layer has its velocity retarded as compared to the free stream velocity due to the effect of viscosity of the fluid, and the next layer has slightly higher velocity, etc. till the free stream velocity is attained at a layer farther away from the surface. The point we are trying to make here is that immediately next to the solid surface, there is essentially a stationary layer of fluid and the heat transfer through this fluid layer is by 'pure conduction'; subsequently, since the next layers of fluid are in motion convection heat transfer occurs.

For this stationary fluid layer, the heat flux is given by Fourier's law:

$$q_{cond} = -k_f\,(dT/dy)\,|_{y=0} \qquad ...(9.3)$$

where k_f is the thermal conductivity of the fluid and $(dT/dy)\,|_{y=0}$ is the temperature gradient at $y = 0$ i.e. at the surface.

9.3 Newton's Law of Cooling and Heat Transfer Coefficient

Governing rate equation for convection heat transfer is given by 'Newton's Law of Cooling' (also known as 'Newton–Rikhman Law'). According to this law, the heat flux in convection heat transfer is given by:

$$q_{conv} = h\cdot(T_s - T_a) \qquad ...(9.4)$$

where h is the convective heat transfer coefficient and $(T_s - T_a)$ is the temperature difference between the hot surface and the flowing fluid. Unit of h is: $W/(m^2C)$ so that the heat flux has units of W/m^2.

Though Eq. 9.4 looks very simple, it is very subtle. The reason is: heat transfer coefficient, h, depends on several factors such as:

(i) the fluid properties like density, viscosity, thermal conductivity and specific heat,
(ii) type of flow (laminar or turbulent),
(iii) shape of fluid passage (circular, rectangle or a flat surface),
(iv) nature of the surface (rough/smooth) and
(v) orientation of the surface

In fact, entire thrust in determining the heat transfer rate in convection is to find out this value of 'h' in a reliable manner.

9.4 Nusselt Number

Since we know that adjacent to the solid surface the fluid layer is stationary and the heat transfer in this fluid layer is by conduction, and the heat transferred by convection subsequently must be equal to this fluid layer, we can equate Eqs. 9.3 and 9.4:

We can write:

$$h = \{-k_f\,(dT/dy)\,|_{y=0}\}/(T_s - T_a) \qquad ...(9.5)$$

i.e. the problem of finding out the value of 'h' reduces to the task of finding out the temperature gradient (dT/dy) at $y = 0$ i.e. at the surface.

Since the heat transfer coefficient depends on flow conditions, its value on a surface varies from point to point. However, we generally take an average value of 'h' by properly averaging the local value of heat transfer coefficient over the entire surface.

It is also common practice to non-dimensionalise the heat transfer coefficient with 'Nusselt number'. Nusselt number is defined as:

$$Nu = \frac{h\cdot\delta}{k_f} \qquad ...(9.5)$$

where δ is a characteristic dimension and k_f is the fluid thermal conductivity.

To get a physical interpretation of the Nusselt number, consider a thin layer of fluid with thickness δ and with a temperature difference of ΔT between the two surfaces. Then, we have:

$$\frac{q_{conv}}{q_{cond}} = \frac{h \cdot \Delta T}{k_f \cdot \frac{\Delta T}{\delta}} = \frac{h \cdot \delta}{k_f} = Nu \quad \text{...(9.6)}$$

In other words, Nusselt number tells us how much the heat transfer is enhanced due to convection as compared to only conduction. Or, higher the Nusselt number, larger the heat transfer by convection. If $Nu = 1$, it means that heat transfer is by conduction alone.

Example 9.1. Air at 25°C flows over a flat surface maintained at 65°C. Temperature measured at a location of 0.3 mm from the surface is 45°C. Find out the value of the local heat transfer coefficient. Thermal conductivity of air at the average temperature may be assumed as 0.027 W/(mC).

Solution.

Data:

$T_a = 25°C \quad T_s = 65°C \quad \delta = 0.0003$ m $\quad \Delta T = 45 - 65 = -20°C \quad k_f = 0.027$ W/(mC)

Now, to find out heat transfer coefficient, apply Eq. 9.5:

$$h = \{-k_f \, (dT/dy)_{y=0}\}/(T_s - T_a) \quad \text{...(9.5)}$$

Now,

$$\frac{dT}{dy} := \frac{-20}{0.0003} \text{ C/m} \quad \textit{(temperature gradient at the surface i.e. at } y = 0)$$

i.e. $$\frac{dT}{dy} = -6.667 \times 10^4 \text{ C/m}$$

(Note that temperature gradient is negative since, starting from the plate surface, as we proceed in the y direction, T decreases as y increases.)

Therefore,

$$h := \frac{0.027 \times 6.667 \times 10^4}{40} \quad \text{...from eqn. (9.5)}$$

i.e. $$h = 45.002 \text{ W/(m}^2\text{C)}.$$

9.5 Velocity Boundary Layer

Concept of 'boundary layer' was introduced by Ludwig Prandtl in the year 1904. According to this concept, when a fluid flows over a surface, the flow field can be considered to be divided into two regions: one, a thin layer adjacent to the solid surface, called the 'boundary layer', where the viscosity effects are predominant and velocity and temperature gradients are very large, and second, a layer beyond the boundary layer where the velocity and temperature gradients are equal to their free stream values. The boundary layer thickness (δ) is arbitrarily defined as that distance from the surface in the y-direction at which the velocity reaches 99% of the free stream velocity, U. Boundary layer concept helps in simplification of momentum equations and, in particular, solution of viscous flow problems was greatly facilitated by this concept.

Let us first study the development of boundary layer for a flow over a flat plate. Flow over a flat plate is important from a practical point of view, since flow over turbine blades and aerofoil sections of air plane wings can be approximated as flow over a flat plate. See Fig. 9.2.

Consider a thin, flat plate. The leading edge and the trailing edge of the plate are shown in the Fig. 9.2. Let a fluid approach the flat plate at a free stream velocity of U. The fluid layer immediately in contact with the plate surface adheres to the surface and remains stationary, and in fluid mechanics, this phenomenon is known as 'no slip' condition. Then, the fluid layer next to this stationary layer has its velocity retarded because of the viscosity effects i.e. due to the frictional force or 'drag' exerted between the stationary and the moving layers. This effect continues with subsequent layers upto some distance in the y-direction till the velocity equals the free stream velocity U. This region of fluid layer in which the viscosity effects are predominant is known as the 'velocity (or hydrodynamic) boundary layer', or simply the 'boundary layer'. Thickness of the boundary layer is arbitrarily defined as that distance in the y-direction from the plate surface at which the velocity is 99% of the free stream velocity. Note the following points in connection with the boundary layer:

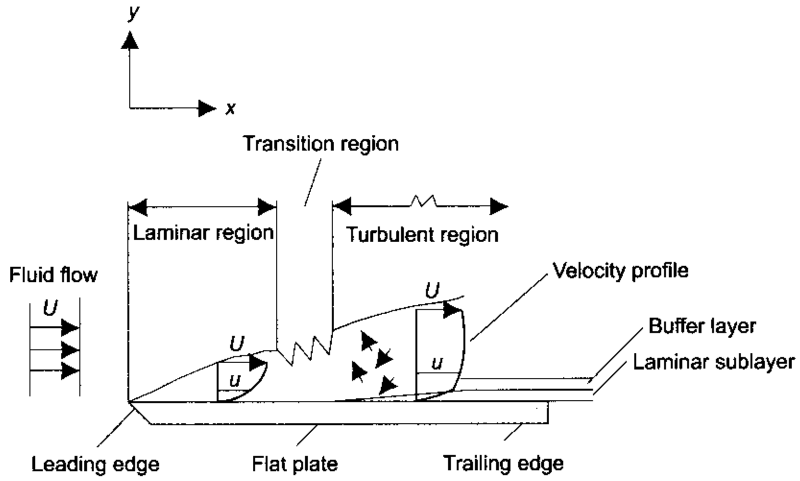

FIGURE 9.2 Development of boundary layer over a flat plate

(i) The boundary layer divides the flow field into two regions: one, 'the boundary layer region' where the viscosity effects are predominant and the velocity gradients are very steep, and, second, 'the inviscid region' where the frictional effects are negligible and the velocity remains essentially constant at the free stream value.

(ii) Since the fluid layers in the boundary layer travel at different velocities, the faster layer exerts a drag force (or frictional force) on the slower layer below it; the drag force per unit area is known as 'shear stress (τ)'. Shear stress is proportional to the velocity gradient at the surface. This is the reason why, in fluid mechanics, the velocity profile has to be found out to determine the frictional force exerted by a fluid on the surface. Shear stress is given by:

$$\tau_s = \mu \cdot \left(\frac{dU}{dy} \right)_{y=0} \quad \text{N/m}^2 \qquad \text{...(9.7)}$$

where μ is 'dynamic viscosity' of the fluid; its unit is kg/(ms) or N.s/m^2. Viscosity is a measure of resistance to flow. For liquids, viscosity decreases as temperature increases, whereas for gases, viscosity increases as the temperature increases. Viscosities of a few fluids at 20°C are given in Table 9.1. It may be observed that viscosity varies by several orders of magnitude for different fluids.

(iii) Use of Eq. 9.7 to determine the surface shear stress is not very convenient, since it requires a mathematical expression for the velocity profile; so, in practice, surface shear stress is determined in terms of the free stream velocity from the following relation:

$$\tau_s = C_f \frac{\rho U^2}{2}, \text{ N/m}^2 \qquad \text{...(9.8)}$$

TABLE 9.1 Dynamic viscosity of a few fluids at 20°C

Fluid	μ(kg/(m.s))
Glycerin	1.49
Engine oil	0.80
Ethyl alcohol	0.00120
Water	0.00106
Freon-12	0.000262
Air	0.0000182

where C_f is a 'friction coefficient' or 'drag coefficient'. ρ is the density of the fluid. C_f is determined experimentally in most cases. Drag coefficient varies along the length of the flat plate. Average value of drag coefficient (C_{fa}) is calculated by suitably integrating the local value over the whole length of the plate and then the drag force over the entire plate surface is determined from:

$$F_D = C_{fa} \cdot A \cdot \frac{\rho U^2}{2}, \text{ N} \quad ...(9.9)$$

where A = surface area, m^2.

(iv) Starting from the leading edge of the plate, for some distance along the length of the plate, the flow in the boundary layer is 'laminar' i.e. the layers of fluid are parallel to each other and the flow proceeds in a systematic, orderly manner. However, after some distance, disturbances appear in the flow and beyond this 'transition region', flow becomes completely chaotic and there is complete mixing of 'chunks' of fluid moving in a random manner i.e. the flow becomes 'turbulent'.

(v) Transition from laminar to turbulent flow depends primarily on the free stream velocity, fluid properties, surface temperature and surface roughness, and is characterized by 'Reynolds number'. Reynolds number is a dimensionless number, defined as:

Re = (Inertia forces/Viscous forces)

Or,

$$Re = \frac{U \cdot x}{\nu} \quad ...(9.10)$$

where U = free stream velocity, m/s

x = characteristic length i.e. for a flat plate it is the length along the plate in the flow direction, from the leading edge, and

ν = kinematic viscosity of fluid = μ/ρ, m^2/s, where ρ is the density of fluid.

When the Reynolds number is low, i.e. when the flow is laminar, inertia forces are small compared to viscous forces and the velocity fluctuations are 'damped out' by the viscosity effects and the layers of fluid flow systematically, parallel to each other. When the Reynolds number is large, i.e. when the flow is turbulent, inertia forces are large compared to the viscous forces and the flow becomes chaotic. For a flat plate, in general, for practical purposes, the 'critical Reynolds number, Re_c' at which the flow changes from laminar to turbulent is taken as 5×10^5. It should be understood clearly that this is not a fixed value but depends on many parameters including the surface roughness.

(vi) There is intense mixing of fluid particles in turbulent region; therefore, heat transfer is more in turbulent flow as compared in laminar flow. This is the reason why special efforts are made in the design of heat exchangers to increase turbulence. However, one has to pay a premium of increased pressure drop i.e. increased power to pump the fluid through the heat exchanger.

(vii) Velocity profile in the laminar flow is approximately parabolic.

(viii) Turbulent region of boundary layer is preceded by transition region as shown in Fig. 9.2.

(ix) Turbulent boundary layer itself is made of three layers: a very thin layer called 'laminar sub-layer', then, a 'buffer layer' and, finally, the 'turbulent layer'.

(x) Velocity profile in the laminar sub-layer is approximately linear, whereas in the turbulent layer the velocity profile is somewhat flat, as shown.

(xi) Thickness of the boundary layer, δ, increases along the flow direction; as we shall see later, δ is related to the Reynolds number as follows: in the laminar flow region:

$$\delta_{\text{lam}} = \frac{5 \cdot x}{(Re_x)^{0.5}} \quad ...(9.11)$$

and for turbulent flow region:

$$\delta_{\text{turb}} = \frac{0.376 \cdot x}{(Re_x)^{0.2}} \quad ...(9.12)$$

where Re_x is the Reynolds number at position x from the leading edge.

9.6 Thermal Boundary Layer

When the temperature of a fluid flowing on a surface is different from that of the surface, a 'thermal boundary layer' develops on the surface, in a manner similar to the development of the velocity boundary layer. Let us illustrate the development of the thermal boundary layer with reference to a flat plate. See Fig. 9.3.

Consider a fluid at an uniform velocity of U and a uniform temperature of T_a approach the leading edge of a thin, flat plate as shown. Let the flat plate be at a uniform temperature of T_s. Let $T_a > T_s$. Then, the first layer that comes in contact with the surface will adhere to the surface (no slip condition) and reach thermal equilibrium with the surface and attain a temperature of T_s. Then, the fluid particles in this layer will exchange energy with the particles in the adjoining layer, which in turn will exchange energy with the subsequent layer, and so on. Thus a temperature profile will develop in the flow field and the temperature will vary from T_s at the surface to T_a at the free stream. In Fig. 9.3, the term $(T - T_s)$ is plotted against y as shown. Thus at the surface, $(T - T_s) = 0$ and at the free stream condition, $(T - T_s) = (T_a - T_s)$. The region in which the temperature variation in the y-direction is significant is known as 'thermal boundary layer'. Thickness of the thermal boundary layer (δ_t) at any location is defined as that distance from the plate surface in the y-direction where $(T - T_s) = 0.99 \times (T_a - T_s)$. i.e where the temperature difference between the fluid and the surface has reached 99% of the maximum possible temperature difference of $(T_a - T_s)$. In other words, at the outer edge of the thermal boundary layer, the dimensionless temperature ratio, $(T - T_s)/(T_a - T_s)$ is equal to 99%.

Thickness of the thermal boundary layer increases with increasing distance along the plate; this is due to the fact that effects of heat transfer are felt more, further downstream.

If the approaching fluid stream temperature T_a is less than the plate surface temperature, then the temperature profile in the thermal boundary layer will be as shown below, in Fig. 9.4:

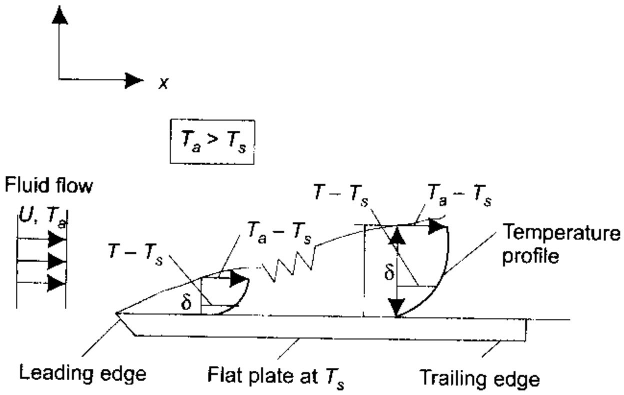

FIGURE 9.3 Development of thermal boundary layer over a flat plate

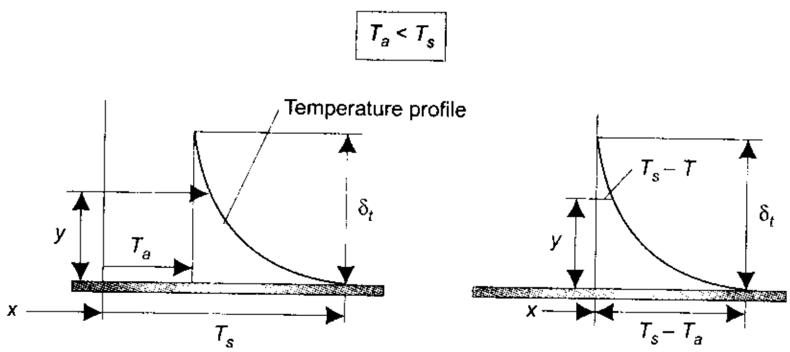

FIGURE 9.4 Thermal boundary layer over a flat plate when $T_a < T_s$

Temperature of the fluid changes from a maximum at the plate surface to the free stream temperature, as we proceed from the surface upwards in the y-direction. Vertical distance from the plate surface where the ratio $(T_s - T)/(T_s - T_a)$ is equal to 99% represents the thickness of the thermal boundary layer.

Velocity profile in the hydrodynamic boundary layer depends on the viscosity of the fluid, whereas temperature profile in the thermal boundary layer depends on the viscosity, specific heat and thermal conductivity of the fluid, in addition to the velocity.

Relative magnitudes of the thicknesses of the hydrodynamic boundary layer (δ) and thermal boundary layer (δ_t) depend on the dimensionless parameter 'Prandtl number' defined as:

Pr = (Molecular diffusivity of momentum)/(Molecular diffusivity of heat)

Or,

$$Pr = \frac{\nu}{\alpha} = \frac{\mu \cdot C_p}{k} \qquad \ldots(9.13)$$

where μ is dynamic viscosity, C_p is the specific heat and k is the thermal conductivity of the fluid.

Also, ν is kinematic viscosity = μ/ρ, and α is the thermal diffusivity.

Prandtl number is of the order of 1 for gases, less than 0.01 for liquid metals and more than 100,000 for heavy oils. See Table 9.2.

TABLE 9.2 Range of Prandtl numbers for fluids

Fluid	*Pr*
Liquid metals	0.004 – 0.030
Gases	0.7 – 1.0
Water	1.7 – 13.7
Light organic fluids	5 – 50
Oils	50 – 100,000
Glycerin	2000 – 100,000

Regarding the relative growth of velocity and thermal boundary layers in a fluid, we may note the following:

(i) For gases, where $Pr = (\nu/\alpha)$ is of the order of 1, we see that the momentum and heat dissipate almost at the same rate i.e. thicknesses of the hydrodynamic and thermal boundary layers are of the same order;

(ii) for liquid metals since $Pr << 1$, it means that heat diffuses at a much higher rate than the momentum for liquid metals i.e. the thermal boundary layer is much thicker than hydrodynamic boundary layer for liquid metals (See Fig 9.5,a), and,

(iii) for heavy oils ($Pr >> 1$), momentum diffuses at a faster rate than heat through the medium and this is evident from Fig. (9.5,b); thus, the thermal boundary layer is much thinner than hydrodynamic boundary layer.

For laminar conditions, thickness of thermal boundary layer is related to hydrodynamic boundary layer, approximately as follows:

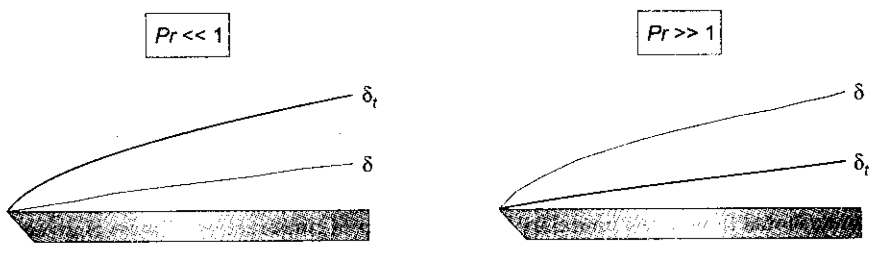

FIGURE 9.5 Thermal and velocity boundary layers over a flat plate for liquid metals and oils

$$\frac{\delta_t}{\delta} = \frac{1}{Pr^{0.33}} \quad \text{...(9.14)}$$

where Pr is the Prandtl number.

9.7 Differential Equations for the Boundary Layer

In convection studies, since there is a fluid flow, we are interested in the shear stress and the friction coefficient; to determine these we need the velocity gradient at the surface. Similarly, to determine the convection coefficient, we need the temperature gradient at the surface. To determine the velocity gradient at the surface, we apply the equation of conservation of momentum (in conjunction with the equation of conservation of mass) to a differential volume element in the boundary layer. And, to determine the temperature gradient at the surface, we apply the equation of conservation of energy to a differential volume element in the boundary layer. We start with the application of equation for conservation of mass:

9.7.1 Conservation of Mass–The Continuity Equation for The Boundary Layer

Consider a differential control volume, of section $(dx.dy)$ and unit depth, within the boundary layer, as shown in Fig. 9.6.

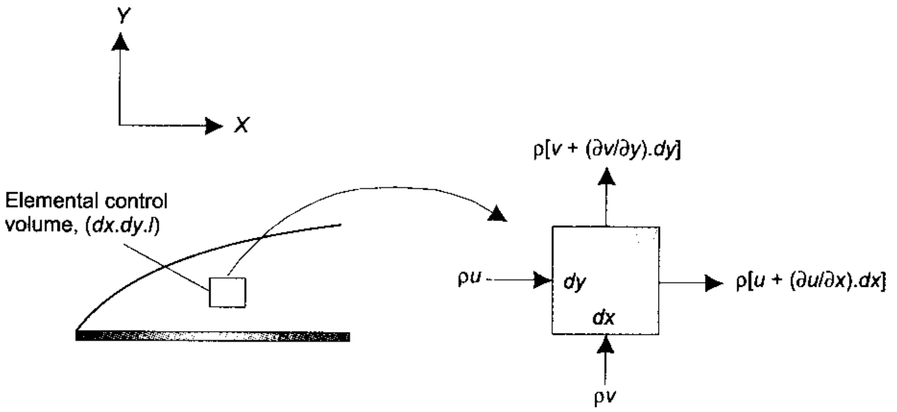

FIGURE 9.6 Elemental control volume in the boundary layer over a flat plate for conservation of mass

Assumptions:

(i) Flow is steady, incompressible
(ii) Constant fluid properties
(iii) Pressure variation is only in the X-direction
(iv) Shear in Y-direction is negligible
(v) Continuity in space and time

Let u and v be the velocity components in the X and Y-directions. Then, remembering that the mass flow rate is given by (density x velocity x area) and that the depth is unity in the Z-direction, we can write:

Mass flow into the control volume in X-direction = $\rho.u.(dy.1)$

Mass flow out of the control volume in X-direction = $\rho.[u + (\partial u/\partial x).dx].(dy.1)$

Therefore, net mass flow into the element in the X-direction = $-\rho.(\partial u/\partial x).dx.dy$

Similarly, net mass flow into the control volume in the Y-direction is = $-\rho.(\partial v/\partial y).dy.dx$

Since the net mass flow into control volume, in steady state, must be equal to zero, we write:

$$-\rho.\{(\partial u/\partial x) + (\partial v/\partial y)\}.dx.dy = 0$$

i.e. for a two-dimensional flow in the boundary layer, equation of conservation of mass is given by:

$$(\partial u/\partial x) + (\partial v/\partial y) = 0 \quad \text{....(9.15)}$$

Eq. 9.15 is known as '**continuity equation**' for two-dimensional, steady flow of an incompressible fluid.

9.7.2 *Conservation of Momentum Equation for The Boundary Layer*

This is obtained by the application of Newton's second law of motion to the differential element, which states that the net force on the element in the X-direction is equal to the net momentum efflux from the control volume in the X-direction. Fig. 9.7 shows the momentum fluxes and forces acting on the differential control volume.

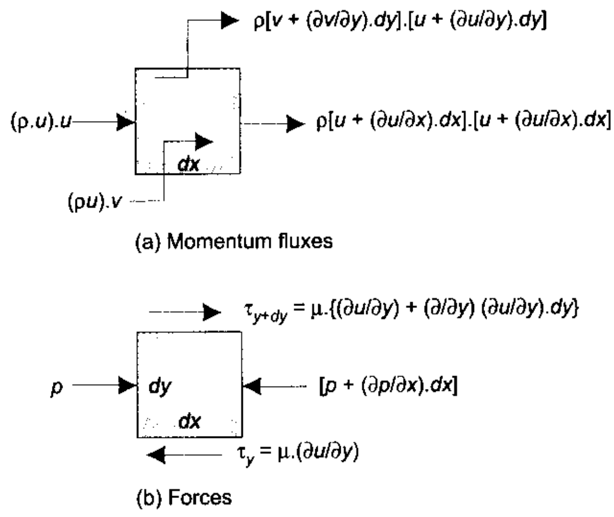

FIGURE 9.7 Conservation of momentum in a two-dimensional, incompressible boundary layer

For no pressure gradients in the Y-direction and with the assumption that viscous shear in the Y-direction is negligible,

Momentum flow in X-direction into left face = $\rho.u^2.dy$

Momentum flow in X-direction out of right face = $\rho.[u + (\partial u/\partial x).dx]^2.dy$

$= \rho.u^2.dy + 2.\rho.u.\ (\partial u/\partial x).dx.dy$

x-momentum flow entering bottom face = $\rho.u.v.dx$

x-momentum flow leaving upper face = $\rho.[v + (\partial v/\partial y).dy].[u + (\partial u/\partial y).dy].dx$

$= \rho.u.v.dx + \rho.u.(\partial v/\partial y).dx.dy + \rho.v.(\partial u/\partial y).dx.dy$

Therefore, net momentum change in the X-direction =

[momentum flux out of the right and top faces] – [momentum flux into the left and bottom faces]

$= [\rho.u^2.dy + 2.\rho.u.(\partial u/\partial x).dx.dy] + [\rho.u.v.dx + \rho.u.(\partial v/\partial y).dx.dy + \rho.v.(\partial u/\partial y).dx.dy]$

$- \rho.u^2.dy - \rho.u.v.dx$

$= 2.\rho.u.(\partial u/\partial x).dx.dy + \rho.u.(\partial v/\partial y).dx.dy + \rho.v.(\partial u/\partial y).dx.dy$

$= \rho.\{u.(\partial u/\partial x) + v.(\partial u/\partial y)\}.dx.dy + \rho.u.\{(\partial u/\partial x) + (\partial v/\partial y)\}.dx.dy$

Now, from continuity Eq. 9.15, we have: $(\partial u/\partial x) + (\partial v/\partial y) = 0$; Therefore, net momentum transfer in the X-direction = $\rho.\{u.(\partial u/\partial x) + v.(\partial u/\partial y)\}.dx.dy$ ...(a)

Now, let us calculate the forces acting on the control volume in the X-direction:

Pressure forces:

Pressure force on the left face is $p.(dy.1)$ and over the right face is

$-[p + (\partial p/\partial x).dx].(dy.1)$

Therefore, net pressure force in the direction of motion is: $-(\partial p/\partial x).dx.dy$

And,

Viscous shear forces:

Viscous shear force at the bottom face is: $\mu(\partial u/\partial y).(dx.1)$

Viscous shear force at the top face is: $[\mu(\partial u/\partial y) + \mu(\partial^2 u/\partial y^2).dy].(dx.1)$

Therefore, net viscous force in the direction of motion =
$[\mu(\partial u/\partial y) + \mu(\partial^2 u/\partial y^2).dy].(dx.1) - \mu(\partial u/\partial y).(dx.1) = \mu(\partial^2 u/\partial y^2).dx.dy$

Therefore,

F_x = Resultant applied force in the X-direction =

Net pressure force in the X-direction + net viscous force in the X-direction

i.e. $$F_x = -(\partial p/\partial x).dx.dy + \mu(\partial^2 u/\partial y^2).dx.dy \quad \ldots(b)$$

Equating Eqs. a and b as per Newton's second law, and neglecting second order differentials, we get:

$$\rho.\{u.(\partial u/\partial x) + v.(\partial u/\partial y)\} = \mu(\partial^2 u/\partial y^2) - (\partial p/\partial x). \quad \ldots(9.16)$$

Eq. 9.16 is known as **'conservation of momentum equation'** for two-dimensional, steady flow of an incompressible fluid.

If the pressure variation in the X-direction is negligible, (which is true for flow over a flat plate since $(\partial U/\partial x) = 0$), Eq. 9.16 reduces to:

$$u.(\partial u/\partial x) + v.(\partial u/\partial y) = \nu.(\partial^2 u/\partial y^2). \quad \ldots(9.17)$$

where $\nu = \mu/\rho$ = kinematic viscosity

9.7.3 Conservation of Energy Equation for The Boundary Layer

Assumptions:

(i) steady, incompressible flow
(ii) conduction is only in the Y-direction
(iii) temperature change in the X-direction is small i.e. negligible conduction in flow direction
(iv) specific heat (C_p) of the fluid is constant
(v) negligible viscous heating
(vi) negligible body forces

Fig. 9.8 shows the rate at which energy is conducted and convected into and out of the differential control volume.

Note that in addition to the conductive terms, there are four convective terms.

Let us write the different energy terms and apply the energy balance which states that net rate of conduction and convection should be equal to zero:

Convective terms:

For the X-direction:

Energy into the control volume = $\rho.C_p.u.T.dy$

Energy out of the control volume = $\rho.C_p.\{u + (\partial u/\partial x).dx\}.\{T + (\partial T/\partial x)\, dx\}.dy$

Therefore, neglecting the product of differentials, net energy convected into the control volume in the X-direction is given by: $-\rho.\ C_p.\{u.(\partial T/\partial x) + T.(\partial u/\partial x)\}.dx.dy$

Similarly, net energy convected into the control volume in the Y-direction is given by:

$$-\rho.C_p.\{v.\ (\partial T/\partial y) + T.(\partial v/\partial y)\}.dx.dy$$

Conductive terms:

Conduction in Y-direction.

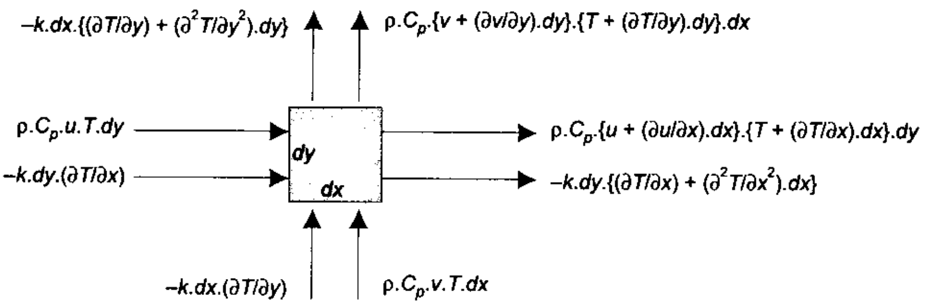

FIGURE 9.8 Conservation of energy in a two-dimensional, incompressible boundary layer

Net conduction into the control volume in the Y-direction is given by:

$$-k.dx.(\partial T/\partial y) - [-k.dx.\{(\partial T/\partial y) + (\partial^2 T/\partial y^2).dy\} = k.(\partial^2 T/\partial y^2).dx.dy$$

Similarly, for completeness, net conduction into the control volume in the X-direction is given by:

$$k.(\partial^2 T/\partial x^2).dx.dy$$

When the viscous work is neglected, making an energy balance, we have:

Algebraic sum total of heat flow to the control volume due to conduction and convection must be equal to zero.

i.e.

$$-\rho.C_p.\{u.(\partial T/\partial x) + T.(\partial u/\partial x)\}.dx.dy - \rho.C_p.\{v.(\partial T/\partial y) + T.(\partial v/\partial y)\}.dx.dy + k.(\partial^2 T/\partial x^2).dx.dy + k.(\partial^2 T/\partial y^2).dx.dy = 0$$

i.e.

$$-\rho.C_p.\{u.(\partial T/\partial x) + T.(\partial u/\partial x) + v.(\partial T/\partial y) + T.(\partial v/\partial y)\}.dx.dy + k.\{(\partial^2 T/\partial x^2) + (\partial^2 T/\partial y^2)\}.dx.dy = 0$$

i.e.

$$-\rho.C_p.\{u.(\partial T/\partial x) + v.(\partial T/\partial y) + T.[(\partial u/\partial x) + (\partial v/\partial y)]\} + k.\{(\partial^2 T/\partial x^2) + (\partial^2 T/\partial y^2)\} = 0$$

Now, from continuity equation, $(\partial u/\partial x) + (\partial v/\partial y) = 0$; also, since the boundary layer is very thin, $(\partial T/\partial y) >> (\partial T/\partial x)$. (i.e. conduction in X-direction is negligible).

Therefore, energy balance equation becomes:

$$u.(\partial T/\partial x) + v.(\partial T/\partial y) = (k/\rho.C_p).(\partial^2 T/\partial y^2)$$

or,

$$u.(\partial T/\partial x) + v.(\partial T/\partial y) = \alpha.(\partial^2 T/\partial y^2), \text{ where } \alpha = \frac{k}{\rho \cdot C_p} = \text{thermal diffusivity} \quad \ldots(9.18)$$

This is the energy equation for a two-dimensional, steady incompressible flow, when the viscous dissipation is neglected, i.e. for very low velocities of flow.

Observe the similarity between Eq. 9.17 for momentum balance and the Eq. 9.18 for energy balance.

In Eq. 9.17, $\nu = \mu/\rho$ = kinematic viscosity, also known as momentum diffusivity. In Eq. 9.18, α is the diffusivity of heat. Their ratio is known as Prandtl number and is equal to:

$$Pr = \nu/\alpha = (\mu/\rho)/(k/\rho.C_p) = C_p.\mu/k \quad \ldots(9.19)$$

If $\nu = \alpha$, then $Pr = 1$ and the momentum and energy equations are identical; thus, Prandtl number controls the relation between the velocity and temperature distributions.

When the viscous dissipation cannot be neglected, as in the case of very viscous fluids (e.g. in journal bearings), or when the fluid shear rate is extremely high, an additional term for 'viscous dissipation, φ' appears on the LHS of the energy balance. φ is given by:

$$\varphi = \mu.\{[(\partial u/\partial y) + (\partial v/\partial x)]^2 + 2.[(\partial u/\partial x)^2 + (\partial v/\partial y)^2] - (2/3).[(\partial u/\partial x) + (\partial v/\partial y)]^2 \quad \ldots(9.20)$$

We shall not consider viscous dissipation in our discussions.

9.8 Methods to Determine Convective Heat Transfer Coefficient

As stated earlier, in convection heat transfer analysis, the primary problem is to determine the heat transfer coefficient. Once this quantity is determined, heat transfer rate from the surface is easily determined by applying Newton's law.

There are generally, five methods available to determine the convective heat transfer coefficient:

(i) dimensional analysis in conjunction with experimental data
(ii) exact mathematical solutions of boundary layer equations
(iii) approximate solutions of boundary layer equations by integral methods
(iv) analogy between heat and momentum transfer, and
(v) numerical analysis

Of course, none of them can by itself, solve all the problems we come across in practice, since each method has its own limitation.

Of the above mentioned methods, 'dimensional analysis' is mathematically simple, but has the disadvantage that it does not give any insight into the phenomenon occurring; also, it does not give any equation that can be solved, but requires experimental data to get the coefficients in the equations. However, this method helps in the interpretation of the experimental data and extends the range of applicability by expressing the data in terms of dimensionless groups.

'Exact solutions of boundary layer equations' involve simultaneous solutions of differential equations derived for the boundary layer. These are rather complicated and solutions are available for a few simple flow situations, such as flow over a flat plate, an airfoil, or a circular cylinder, in laminar flow. Describing the turbulent flow mathematically is rather difficult. We shall only give an outline of this method, since our emphasis is on practical solutions to convection heat transfer problems by using empirical relations.

'Approximate solutions for boundary layer equations' consider a finite control volume for analysis, rather than an infinitesimal control volume, and integral equations are derived; however, solution requires assuming equations to describe the velocity and temperature profiles satisfying the boundary conditions. This method is relatively simple, and it is possible to get solutions to problems that cannot be treated by exact method of analysis. This method can be applied to turbulent flow also.

'Analogy between heat and momentum transfer' is a very useful tool to deduce the convective heat transfer coefficient by the knowledge of flow friction data only, particularly for turbulent flows, without actually conducting heat transfer experiments. This method utilizes the fact that the momentum and energy equations have the same form, under certain conditions, and therefore, the solutions also must have the same form. Further, it is simple to conduct flow (friction) experiments, as compared to heat transfer experiments.

'Numerical methods' involves discretizing the differential equations and are therefore approximate. Solutions are obtained at discrete points in time and space rather than continuously; however, accuracy can be improved to acceptable levels by taking sufficiently close grids. Main advantage of numerical methods is that variation in fluid properties and boundary conditions can be easily handled.

9.8.1 Dimensional Analysis

Dimensional analysis considers the various quantities that contribute to the phenomenon and reduces these variables into dimensionless groups; however, dimensional analysis alone is not of much use, and this method must always be supplemented by experimental data since to determine the coefficients in the functional relationships between the dimensionless groups we need actual, practical data. Also, it is necessary to have some insight into the problem before we start the analysis, since we have to first list the pertinent variables that influence the phenomenon. Once this is done, mathematics involved is minimum, and the method can be applied routinely to most of the problems.

9.8.1.1 Primary dimensions and dimensional formulas. Fundamental axiom of dimensional analysis is that equations describing a physical phenomenon must be dimensionally homogeneous (i.e. dimensions of the two sides of the equation are identical) and units therein must be consistent.

'Dimension' is a qualitative expression whereas unit is quantitative. For example, when the distance between two points is spoken of as 'length' it is qualitative; instead, if we say that the distance is so many metres or kilometres or miles, we are speaking in terms of 'Units'.

In S.I. system, there are four 'primary dimensions' viz. Length (L), Mass (M), Time (t) and Temperature (T). Other derived quantities can be expressed in terms of these primary dimensions. Dimensional formula for a physical quantity is obtained from its definition or from physical laws involved. For example,

Dimension of length of a bar: [L]

Dimension of velocity: Distance/time: $[L/t] = L.t^{-1}$

Dimension of Force: Mass x acceleration = $[M.L/t^2] = [M.\ L.\ t^{-2}]$

Dimension of Work : Force x distance : $[M.\ L.\ t^{-2}].L = [M.\ L^2.\ t^{-2}]$

Dimension of Power : Work/time : = $[M.\ L^2.t^{-3}]$, ..., etc.

Table 9.3 shows a few physical quantities, their symbols, units and dimensional formulas.

9.8.1.2 Buckingham π theorem. This theorem is used to determine the number of independent dimensionless groups that can be obtained from a set of physical quantities that govern a given phenomenon.

According to this rule, if the number of pertinent physical variables governing a phenomenon is 'n', and the number of primary dimensions to express the dimensional formulas of these n quantities is 'm', then, the number of independent dimensionless groups that can be formed by combining these physical quantities is given by $(n - m)$. If these dimensionless groups are designated by $\pi_1, \pi_2, \ldots$, etc. then, relation between them can be expressed as:

$$F(\pi_1, \pi_2, \pi_3, \ldots) = 0 \qquad \ldots(9.21)$$

TABLE 9.3 Some physical quantities of importance in heat transfer and their dimensions

Quantity	Symbol	Units	Dimensions
Mass	M	kg	M
Length	L, d, D, x	m	L
Time	t, τ	s	t
Temperature	T, θ	K	T
Area	A	m^2	L^2
Volume	V	m^3	L^3
Velocity	V, U	m/s	Lt^{-1}
Acceleration	a	m/s^2	Lt^{-2}
Force	F	N	$M\,Lt^{-2}$
Work	W	Nm(=J)	$M\,L^2t^{-2}$
Power	P	W(=J/s)	$M\,L^2t^{-3}$
Density	ρ	kg/m^3	ML^{-3}
Pressure, stress	p, σ	N/m^2	$M\,L^{-1}t^{-2}$
Viscosity	μ	kg/(ms)	$M\,L^{-1}t^{-1}$
Kinematic viscosity	$\nu(=\mu/\rho)$	m^2/s	L^2t^{-1}
Specific heat	c_p	J/kgK	$L^2t^{-2}T^{-1}$
Thermal conductivity	k	W/(mK)	$MLt^{-3}T^{-1}$
Thermal diffusivity	α	m^2/s	L^2t^{-1}
Heat transfer coefficient	h	$W/(m^2K)$	$Mt^{-3}T^{-1}$
Coefficient of volume expansion	β	1/K	T^{-1}

For example, if in a problem, there are 5 physical quantities which are described by 3 primary dimensions only, then there are (5 – 3) = 2 dimensionless groups and the solution is of the form:

$$F(\pi_1, \pi_2) = 0. \quad \ldots(9.22)$$

Or,

$$\pi_1 = f(\pi_2) \quad \ldots(9.23)$$

Then experimental data can be presented by plotting π_1 against π_2.

If there are 3 dimensionless groups in another problem, the solution is of the form:

$$F(\pi_1, \pi_2, \pi_3) = 0. \quad \ldots(9.24)$$

Or,

$$\pi_1 = f(\pi_2, \pi_3) \quad \ldots(9.25)$$

Now, experimental data can be presented by plotting π_1 against π_2 for different values of π_3.

While applying the Buckingham method, after determining the number of π terms that can be formed, further procedure is as follows: Of the total of 'n' number of variables, select a 'core group' of 'm' number of variables, which repeat for each π term; these are known as 'repeated variables'; then, each π term is formed by the core group plus one of the remaining $(n - m)$ variables. Each of the variables in the core group is raised to a suitable power to maintain dimensional homogeneity. Selection of the core group should be done as per the following thumb rules:

(a) variables in the core group must contain among themselves all the fundamental dimensions involved in the phenomenon.

(b) the repeating variables must not form dimensionless groups among themselves

(c) invariably, dependent variable should not be incuded in the core group

(d) no two variables in the core group should have the same dimensions

(e) in general, repeating variables should be chosen such that one variable contains a geometric property (e.g. length 'l', diameter 'D' or height 'h'), other variable contains a flow property (e.g. velocity 'V', acceleration 'a' etc.), and the other variable contains a fluid property (e.g. density 'ρ', dynamic viscosity 'μ' etc.). In most of the cases, repeated variables or the core group consist of : (l, V, ρ), (d, V, ρ), (l, V, μ), or (d, V, μ).

Procedure of applying the Buckingham method is illustrated below:

9.8.1.3 Dimensional analysis for forced convection. Now let us illustrate the application of Buckigham's π theorem to the case of convection heat transfer for a fluid flowing across a heated tube; of course, same approach is applicable for heat transfer for a fluid flowing inside a tube or flowing over a plate.

First, it is necessary to list the pertinent parameters influencing the physical phenomenon. From the description of the problem, it appears reasonable to assume that the physical quantities listed below (along with their dimensional formulas) are relevant to this problem:

1. Tube diameter (D)...[L]
2. Fluid density (ρ)...[ML^{-3}]
3. Fluid velocity (V)...[Lt^{-1}]
4. Fluid viscosity (μ)...[$ML^{-1}t^{-1}$]
5. Specific heat (C_p)...[$L^2t^{-2}T^{-1}$]
6. Thermal conductivity (k)...[$MLt^{-3}T^{-1}$], and
7. Heat transfer coefficient (h)...[$Mt^{-3}T^{-1}$]

Thus, we see that there are 7 pertinent variables affecting the physical phenomenon and they contain 4 fundamental dimensions L, M, t and T.

Then, from Buckingham's π theorem, we deduce that (7 – 4) = 3 independent dimensionless groups would be formed to correlate experimental data.

Now, let us form the 'core group' of 4 variables, keeping in mind the principles enumerated above. Let us choose d, V, ρ, and h for the core group. They contain among themselves all the primary dimensions; they do not form dimensionless groups among themselves; no two variables have same dimensions; and, one variable(D) is a geometric property, one variable(V) is a flow property, and ρ is a fluid property. Then, the different π terms are obtained by combining the core group with each one of the remaining (7 – 4) properties:

$$\pi_1 = h^a.\rho^b.D^c.V^d.\mu$$
$$\pi_2 = h^m.\rho^n.D^p.V^q.C_p$$
$$\pi_3 = h^w.\rho^x.D^y.V^z.k$$

Exponents of terms in π-terms are chosen so as to make the π terms dimensionless. So, we start with π_1 and write the dimensional formulas of each quantity and apply the requirement of dimensional homogeneity:

For π_1:

$$\pi_1 = h^a.\rho^b.D^c.V^d.\mu$$
$$M^0L^0t^0T^0 = 1 = [Mt^{-3}T^{-1}]^a.\ [ML^{-3}]^b.[L]^c.\ [Lt^{-1}]^d.\ [ML^{-1}t^{-1}]$$

Equating the exponents of M, L, t and T on either side, for dimensional homogeneity:

Exponents of M: $0 = a + b + 1$

Exponents of L: $0 = -3b + c + d - 1$

Exponents of t: $0 = -3a - d - 1$

Exponents of T: $0 = -a$

Solving the above equations, we get:

$$a = 0;\ b = -1;\ c = -1;\ d = -1$$

Therefore,

$$\pi_1 = \rho^{-1}.D^{-1}.V^{-1}.\mu$$

i.e.

$$\pi_1 = \mu/(\rho.V.D)$$

Since π_1 is dimensionless anyway,

we shall choose: $\pi_1 = \rho.V.D/\mu$

Recognize that π_1 is the dimensionless **Reynolds number (*Re*).**

For π_2:

$$\pi_2 = h^m.\rho^n.D^p.V^q.C_p$$

Then, $M^0L^0t^0T^0 = 1 = [Mt^{-3}T^{-1}]^m.\ [ML^{-3}]^n.[L]^p.\ [Lt^{-1}]^q.\ [L^2t^{-2}T^{-1}]$

Equating the exponents of M, L, t and T on either side, for dimensional homogeneity:

Exponents of M: $0 = m + n$

Exponents of L: $0 = -3n + p + q + 2$

Exponents of t: $0 = -3m - q - 2$

Exponents of T: $0 = -m - 1$

Solving the above equations, we get:

$$m = -1; n = 1; p = 0; q = 1$$

Therefore,

$$\pi_2 = h^{-1}.\rho.V.C_p$$

i.e.

$$\pi_2 = (C_p.\rho.V)/h$$

Since the dimensions of h and k/D are same, we write:

$$\pi_2 = (C_p.\rho.V.D)/k$$

Dividing this by another dimensionless number, i.e. Reynolds number gives again another dimensionless number; so, we get:

$$\pi_2 = \{(C_p.\rho.V.D)/k\}/\{\rho.V.D/\mu\} = \mu.C_p/k$$

Recognise that π_2 is the dimensionless **Prandtl number (*Pr*).**

For π_3:

$$\pi_3 = h^w.\rho^x.D^y.V^z.k$$

Then,

$$M^0L^0t^0T^0 = 1 = [Mt^{-3}T^{-1}]^w.\ [ML^{-3}]^x.[L]^y.\ [Lt^{-1}]^z.\ [MLt^{-3}T^{-1}]$$

Equating the exponents of M, L, t and T on either side, for dimensional homogeneity:

Exponents of M: $0 = w + x + 1$

Exponents of L: $0 = -3x + y + z + 1$

Exponents of t: $0 = -3w - z - 3$

Exponents of T: $0 = -w - 1$

Solving the above equations, we get:

$$w = -1; x = 0; y = -1; z = 0$$

Therefore,

$$\pi_3 = h^{-1}.D^{-1}.k = k/(h.D)$$

Since $k/(h.D)$ is dimensionless, $(h.D)/k$ is also dimensionless. So, we choose:

$$\pi_3 = (h.D)/k$$

Recognize that π_3 is the dimensionless **Nusselt number (*Nu*).**

Then, according to the Buckingham π theorem,

$$\pi_3 = F(\pi_1, \pi_2)$$

Or,

$$Nu = C.\ Re^m.Pr^n. \quad ...(9.26)$$

where C, m and n are constants evaluated experimentally.

Eq. 9.26 is the desired relation among the various physical quantities affecting forced convection across a tube, expressed in terms of dimensionless numbers *Nu*, *Re* and *Pr*.

Note:

(a) If we had taken (D, ρ, μ, k) for the core group (or, repeating variables), then combining the core group with V, c_p and h in turn, we would have got, respectively:

$$\pi_1 = (\rho\ V\ D)/\mu = Re$$

$$\pi_2 = \mu C_p/k = Pr, \text{ and}$$

$$\pi_3 = h\ D/k = Nu$$

i.e. the same result as obtained earlier.

(b) If, instead, we choose (V, μ, ρ, C_p) as the core group, then the dimensionless terms obtained are:

$$Re = (\rho\ V\ D)/\mu$$

$$Pr = \mu C_p/k, \text{ and}$$

$$St = h/(\rho\ V\ C_p) = h/(G\ C_p) = \textbf{Stanton number,}$$

where $G = \rho.V$ = mass velocity

In fact, another way of expressing heat transfer correlations is:

$$St = F(Re, Pr). \quad ...(9.27)$$

9.8.1.4 Advantages and limitations of dimensional analysis

Advantages:

(i) It is mathematically quite simple.

(ii) When a given physical phenomenon depends on a large number of variables, dimensional analysis reduces the number of variables for experimentation by getting the dimensionless numbers with suitable combination of those variables. Advantage of having lesser number of variables for experimentation is obvious.

(iii) Dimensional analysis helps in interpretation of experimental data and in deriving suitable empirical, design equations.

(iv) It also helps in planning the experimental work for a particular problem.

(v) It helps to extend the range of experimental results; for example, if a particular set of results for air in forced convection is expressed in terms of Nusselts number, Reynolds number and Prandtl numbers, then the same results can be applied to another fluid, say, water, if the corresponding dimensionless numbers are the same.

(vi) It helps in getting a partial solution to problems, when the mathematical solution is too complicated.

Limitations:

(i) It does not give any insight into the physical phenomenon occurring.

(ii) Selection of variables has to be done with care; if it is wrongly done, results will be erroneous.

(iii) It does not give an exact functional relation which can be solved; dimensional analysis requires experimental data to get the coefficients in the functional relationship.

(iv) If it is required to get the effect of one particular variable on the rest of the variables in a particular problem, it is difficult to get this information by dimensional analysis.

Application of dimensional analysis to the case of heat transfer by natural convection will be described in the next chapter.

9.8.1.5 Dimensionless numbers and their physical significance. There are many dimensionless numbers that we come across in heat transfer studies. Their physical significance must be clearly understood and this is is facilitated by expressing these dimensionless numbers as the ratios of two forces. This requires a little explanation:

Many times, in fluid mechanics and heat transfer studies, it becomes impossible or impracticable to conduct experiments on the actual prototype size of the system. Then, studies are done on a model of reduced size. Then, the question arises as to how to relate the results of the experiments done on the model to the actual prototype. To be able to do so, certain criteria have to be satisfied. These criteria, known as 'criteria for similitude' are the following:

(a) **Geometrical similarity** Two objects are geometrically similar if the ratios of corresponding linear dimensions are equal.

(b) **Kinematic similarity** This represents similarity of motion , i.e. if the ratios of velocities of corresponding particles are equal, there is said to be kinematic similarity.

(c) **Dynamic similarity** This represents similarity of forces. If there is kinematic similarity and in addition, the ratios of homologous forces in the systems are also the same, there is said to be dynamic similarity.

If all the above criteria are satisfied, then there is complete correspondence or similarity between the model and the prototype.

Further, in an incompressible flow, if the conditions of geometrical similarity and dynamic similarity are satisfied, then kinematic similarity is automatically achieved.

Geometric similarity can be easily achieved by constructing the model of the actual system to a certain reduced scale. One way of ensuring dynamic similarity is by making sure that some relevant dimensionless numbers are the same for both the model and the prototype, since these dimensionless numbers can be expressed as ratios of certain forces. Let us illustrate this by considering different forces that are relevant to fluid mechanics and heat transfer:

(1) Inertia force (F_i):

F_i = mass x acceleration , i.e.

$$F_i = \rho \cdot L^3 \cdot \frac{dV}{dt} = \frac{\rho \cdot L^3 \cdot V}{\frac{L}{V}} = \rho \cdot L^2 \cdot V^2 \qquad ...(a)$$

(2) Viscous force (F_v):

F_v = shear stress x area, i.e.

$$F_v = \tau \cdot L^2 = \mu \cdot \frac{dV}{dy} \cdot L^2 = \mu \cdot V \cdot L \qquad \text{...(b)}$$

(3) Gravity force (F_g):

F_g = mass x gravitational acceleration, i.e.

$$F_g = \rho \cdot L^3 \cdot g \qquad \text{...(c)}$$

(4) Surface tension (F_t):

$$F_t = \sigma \cdot L \qquad \text{...(d)}$$

where σ is the coefficient of surface tension (units: Force/unit length)

(5) Elasticity force (F_e):

$$F_e = E_v \cdot L^2 \qquad \text{...(e)}$$

where E_v is the bulk modulus of elasticity of the fluid.

(6) Pressure force (F_p):

F_p = pressure x area, i.e.

$$F_p = \Delta p \cdot L^2 \qquad \text{...(f)}$$

Now, let us form the ratios of inertia force with other forces:

Ratios of forces:

(a) $$\frac{\text{Inertia force}}{\text{Viscous force}} = \frac{\rho \cdot V^2 \cdot L^2}{\mu \cdot V \cdot L} = \frac{\rho \cdot V \cdot L}{\mu} = \text{Reynolds Number} = Re$$

(b) $$\frac{\text{Inertia force}}{\text{Gravity force}} = \frac{\rho \cdot V^2 \cdot L^2}{\rho \cdot g \cdot L^3} = \frac{V^2}{g \cdot L} = \left(\frac{V}{\sqrt{g \cdot L}}\right)^2 = (Fr)^2$$

Here, Fr is known as 'Froude number'.

(c) $$\frac{\text{Inertia force}}{\text{Surface Tension force}} = \frac{\rho \cdot V^2 \cdot L^2}{\sigma \cdot L} = \frac{\rho \cdot V^2 \cdot L}{\sigma} = \text{Weber Number} = Wn$$

(d) $$\frac{\text{Inertia force}}{\text{Elasticity force}} = \frac{\rho \cdot V^2 \cdot L^2}{E_v \cdot L^2} = \frac{V^2}{\frac{E_v}{\rho}} = \frac{V^2}{V_s^2} = (Ma)^2$$

where $V_s = \sqrt{\frac{E_v}{\rho}}$ = Sonic velocity and,

Ma = Mach number

(e) $$\frac{\text{Inertia force}}{\text{Pressure force}} = \frac{\rho \cdot V^2 \cdot L^2}{\Delta P \cdot L^2} = \frac{\rho \cdot V^2}{\Delta P} = \text{Euler Number} = En$$

Dimensionless numbers mentioned above occur frequently in fluid mechanics. Some of the dimensionless numbers occurring in heat transfer are:

Reynolds number:

We have:

$$Re = \frac{\rho \cdot V \cdot L}{\mu} = \frac{\rho \cdot V^2 \cdot L^2}{\mu \cdot V \cdot L}$$

i.e.

$$Re = \text{Inertia force/Viscous force}$$

i.e. Reynolds number is a measure of relative magnitudes of inertial and viscous forces occurring in a given flow situation. At low velocities, Reynolds number is low, i.e. viscous effects are large and any flow disturbances are easily damped by viscous effects and the different layers in the flow move systematically, parallel to each other; this is called laminar flow. If, on the other hand, the Reynolds number is large, effect of inertial forces are predominant and the flow pattern is completely random, with the chunks of particles moving in all directions; this is called turbulent flow. Thus, Reynolds number denotes the type of flow i.e. if the flow is laminar or turbulent.

Prandtl number:
We have:

$$Pr = \frac{\mu \cdot C_p}{k} = \frac{\rho \cdot \nu \cdot C_p}{k} = \frac{\nu}{\left(\frac{k}{\rho \cdot C_p}\right)} = \frac{\nu}{\alpha} = \frac{\text{kinematic viscosity}}{\text{thermal diffusivity}}$$

i.e. Prandtl number is the ratio of kinematic viscosity of the fluid to its thermal diffusivity. ν represents the diffusion of momentum through the fluid whereas α represents the diffusion of heat (energy) through the fluid. Therefore Pr is a measure of relative effectiveness of momentum and energy transport in the medium by diffusion. For oils $Pr >> 1$, and this signifies that in oils, momentum transport is more rapid than the transport of energy; for gases, $Pr \sim 1$ and this means that in gases, momentum and energy are transported by diffusion at almost the same rate. For the case of liquid metals, where $Pr << 1$, the energy transport is many times more rapid as compared to the transport of momentum.

Prandtl number is also the significant parameter which influences the relative growth of velocity and temperature profiles. Hydrodynamic and thermal boundary layer thicknesses are related by:

$$\frac{\delta}{\delta_t} = Pr^n$$

where 'n' is a positive exponent.

For gases ($Pr \sim 1$), $\delta \sim \delta_t$, for oils ($Pr >> 1$), $\delta >> \delta_t$, and for liquid metals ($Pr << 1$), $\delta << \delta_t$.

Nusselt number:
We have:

$$Nu = \frac{h \cdot L}{k}$$

Consider a plate at a temperature T_s, over which a fluid at a temperature T_a is flowing; then, immediately adjacent to the surface there will be a stationary layer of fluid. In this layer, heat transfer is, obviously by conduction and then the heat is transferred to the stream by convection. Making an energy balance and equating these two quantities,

$$Q = -k \cdot A \cdot \left(\frac{dT}{dy}\right)_{y=0} = h \cdot A \cdot (T_s - T_a)$$

i.e.

$$h = \frac{-k \cdot \left(\frac{dT}{dy}\right)_{y=0}}{T_s - T_a}$$

i.e.

$$\frac{h \cdot L}{k} = \frac{-\left(\frac{dT}{dy}\right)_{y=0}}{\frac{(T_s - T_a)}{L}}$$

i.e. Nusselt number may be interpreted as a ratio of temperature gradient at the surface to an overall, reference temperature gradient.

Looking at it in another way, multiplying both the numerator and denominator of the expression for Nu by ΔT, we can write:

$$Nu = \frac{h \cdot L}{k} = \frac{h \cdot \Delta T}{k \cdot \frac{\Delta T}{L}} = \frac{\text{convective heat flux}}{\text{conductive heat flux}}$$

i.e. Nusselts number is an indication of the enhancement of heat transfer by convection.

Stanton number:
We have:

$$St = \frac{h}{G \cdot C_p} = \frac{h}{\rho \cdot V \cdot C_p}$$

This can be written as:

$$St = \frac{h}{G \cdot C_p} = \frac{h}{\rho \cdot V \cdot C_p} = \frac{\frac{h \cdot L}{k}}{\left(\frac{\rho \cdot V \cdot L}{\mu}\right) \cdot \left(\frac{\mu \cdot C_p}{k}\right)} = \frac{Nu}{Re \cdot Pr}$$

Stanton number is expressed in terms of other three dimensionless numbers, namely Nusselts number, Reynolds number and Prandtl number. Note that Stanton number comes into picture only in connection with the forced convection heat transfer, since the term for velocity (V) is contained in the expression for Nu.

In another interpretation, if the temperature difference between the wall surface and the bulk of the fluid is ΔT,

convective heat flux = $h.\Delta T$; and

energy transported by the bulk fluid flow per unit cross-section of flow area =

mass flow rate x C_p x ΔT = $(V.\rho).C_p.\Delta T$

Therefore, taking their ratio:

$$\frac{h \cdot \Delta T}{(V \cdot \rho) \cdot C_p \cdot \Delta T} = \frac{h}{V \cdot \rho \cdot C_p} = Nu$$

In other words, Nusselt number may also be interpreted as the ratio of convective heat flux to the rate of energy transport by the bulk flow of the fluid per unit area of flow cross-section.

Peclet number:
We have:

$$Pe = \frac{\rho \cdot V \cdot L \cdot C_p}{k} = \frac{\rho \cdot V \cdot L}{\mu} \cdot \frac{C_p \cdot \mu}{k} = Re \cdot Pr$$

i.e. Peclet number may be expressed as the product of Reynolds number and Prandtl numbers. Again, as we have shown above, energy transported by the bulk fluid flow per unit cross-section of flow area =

mass flow rate x C_p x ΔT = $(V.\rho)$.Cp.ΔT, and

heat flux due to conduction across a distance L for the same $\Delta T = k\Delta T/L$. Taking their ratio:

$$\frac{(V \cdot \rho) \cdot C_p \cdot \Delta T}{\left(\frac{k \cdot \Delta T}{L}\right)} = \frac{\rho \cdot V \cdot L \cdot C_p}{k} = Pe$$

i.e. Peclet number may be interpreted as the ratio of rate of heat transfer by bulk flow to the rate of heat transfer by conduction.

Graetz number:
This dimensionless number is related to the heat transfer to a fluid flowing through a circular pipe. By definition, it is the ratio of heat capacity of the fluid flowing per unit length of the pipe to the thermal conductivity of the pipe i.e.

$$Gz = \frac{\left(\frac{m \cdot C_p}{L}\right)}{k} = \frac{\frac{\pi \cdot D^2}{4} \cdot \rho \cdot V \cdot C_p}{k \cdot L} = \frac{\pi}{4} \cdot \frac{\rho \cdot V \cdot D}{\mu} \cdot \frac{\mu \cdot C_p}{k} \cdot \frac{D}{L} = \frac{\pi}{4} \cdot Re \cdot Pr \cdot \frac{D}{L}$$

where D is the diameter and L is the length of pipe. Therefore, Graetz number is similar to Peclet number, but is used in connection with heat transfer analysis of laminar flow in pipes.

Grashoff number:

Grashoff number occurs only in connection with heat transfer in natural convection (we shall study this later). We have, by definition:

$$Gr = \frac{L^3 \cdot \rho^2 \cdot \beta \cdot g \cdot \Delta T}{\mu^2}$$

This can be written as:

$$Gr = \frac{L^3 \cdot \rho^2 \cdot \beta \cdot g \cdot \Delta T}{\mu^2} = (\rho \cdot L^3 \cdot \beta \cdot g \cdot \Delta T) \cdot \frac{\rho}{\mu^2} = (\rho \cdot L^3 \cdot \beta \cdot g \cdot \Delta T) \cdot \left[\frac{\rho \cdot V^2 \cdot L^2}{(\mu \cdot V \cdot L)^2}\right]$$

In other words,

$$G_1 = \text{Buoyant force} \cdot \frac{\text{Inertia force}}{(\text{Viscous force})^2}$$

Role of Grashoff number in natural (free) convection is similar to that of Reynolds number in forced convection.

9.8.2 *Exact Solutions of Boundary Layer Equations*

Here, we shall illustrate the method in connection with the heat transfer for flow on a flat plate. However, we shall only give an outline of the method, since, as we stated earlier, our focus is to enumerate the empirical relations useful for practical calculations.

Recollect that the equations of continuity, momentum and energy for the boundary layer on a flat plate are given, respectively, by:

$$(\partial u/\partial x) + (\partial v/\partial y) = 0 \quad \text{...(9.15)}$$

$$u.(\partial u/\partial x) + v.(\partial u/\partial y) = \nu.(\partial^2 u/\partial y^2) \quad \text{...(9.17)}$$

$$u.(\partial T/\partial x) + v.(\partial T/\partial y) = \alpha.(\partial^2 T/\partial y^2) \quad \text{...(9.18)}$$

Now, solving the momentum equation in conjunction with the continuity equation gives the velocity distribution, boundary layer thickness and shear stress (or friction force) at the surface. Exact mathematical solution is rather complex; its outline is given below:

Since the velocity profiles at different distances from the leading edge of the plate are similar, they can be considered to differ from each other only by a 'stretching factor' in the y-direction. So, the dimensionless velocity u/U can be expressed at any location x as a function of dimensionless distance y/δ from the wall.

Define:

$$\eta = y \cdot \sqrt{\frac{U}{\nu \cdot x}} = \text{stretching factor}$$

Also, a stream function $\psi(x, y)$ is defined such that it satisfies the continuity equation, and letting

$$\psi = \sqrt{\nu \cdot x \cdot U} \cdot f(\eta) \text{ where } u = \partial\psi/\partial y; \quad v = \partial\psi/\partial x$$

Substituting for the terms in the momentum equation in terms of η gives an ordinary, nonlinear, third order differential equation:

$$f(\eta) \cdot \frac{d^2 f(\eta)}{d\eta^2} + 2 \cdot \frac{d^3 f(\eta)}{d\eta^3} = 0$$

Solution of this differential equation was obtained numerically, by Blasius. The result is shown in Fig. 9.9.

In Fig. 9.9, abscissa is a dimensionless distance $(y/x).Re_x^{0.5}$ and the ordinate is a dimensionless velocity (u/U), where u is the local velocity in the x-direction and U is the free stream velocity.

Two important observations are to be made from Fig. 9.9:

(a) first, when the x-coordinate reaches a value of 5 , the y-coordinate is 0.99 i.e. the local velocity reaches 99% of the stream velocity value when $(y/x).Re_x^{0.5}$ reaches a value of 5. However, from the definition of the boundary layer thickness δ, we know that $y = \delta$ when $u/U = 99\%$. Therefore, we can write:

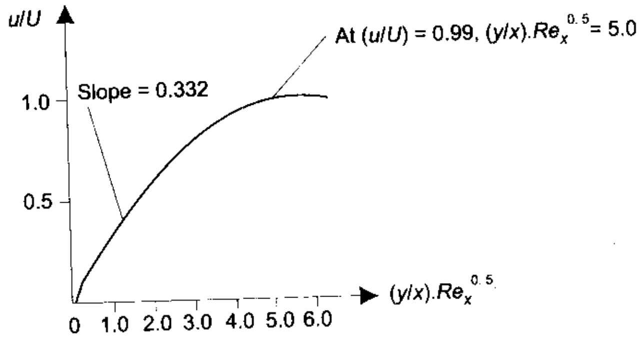

FIGURE 9.9 Velocity ratio in laminar boundary layer, as per Blasius

$$\delta = \frac{5 \cdot x}{\sqrt{Re_x}} \qquad ...(9.28)$$

where $Re_x = \frac{\rho \cdot U \cdot x}{\mu}$, local value of Reynolds number.

(b) second observation is that the slope at $y = 0$ is 0.332, i.e.

$$\left[\frac{d\left(\frac{u}{U}\right)}{d\left(\frac{y}{x} \cdot \sqrt{Re_x}\right)} \right]_{y=0} = 0.332$$

We get:

$$\left(\frac{du}{dy}\right)_{y=0} = 0.332 \cdot \frac{U}{x} \cdot \sqrt{Re_x} \qquad ...(9.29)$$

Then, the wall shear stress, τ is given by:

$$\tau = \mu \cdot \left(\frac{du}{dy}\right)_{y=0} = 0.332 \cdot \mu \cdot \frac{U}{x} \cdot \sqrt{Re_x} \qquad ...(9.30)$$

And, the friction coefficient (or drag coefficient), is by definition:

$$C_{fx} = \frac{\tau}{\left(\frac{\rho \cdot U^2}{2}\right)} = \frac{0.664}{\sqrt{Re_x}} \qquad ...(9.31)$$

This is the local value of friction coefficient.

Average value of friction coefficient (C_{fa}) over a plate length of L is obtained by integrating Eq. 9.31 between $x = 0$ and $x = L$. i.e.

$$C_{fa} = \frac{1}{L} \cdot \int_0^L C_{fx}\, dx = \frac{1.328}{\sqrt{Re_L}} \qquad ...(9.32)$$

i.e.

$$C_{fa} = 2 \cdot C_{fL} \qquad ...(9.33)$$

Thus, for laminar flow over a flat plate, average friction coefficient is twice the value of local friction coefficient at $x = L$.

Solution of the energy Eq. 9.18 gives the value of convective heat transfer coefficient.

Observe the similarity between the equation of momentum 9.17 and equation of energy 9.18. This fact led Pohlhausen to follow Blasius assumption of a similarity parameter and stream function as follows:

$$\eta = y \cdot \sqrt{\frac{U}{\nu \cdot x}} = \text{similarity parameter}$$

$$\psi = \sqrt{\nu \cdot x \cdot U} \cdot f(\eta)$$

and, the following ordinary differential equation is obtained:

$$\frac{d^2\theta}{d\eta^2} + \frac{Pr}{2} \cdot f \cdot \frac{d\theta}{d\eta} = \frac{T - T_s}{T_a - T_s}$$

where $$\theta = \frac{T - T_s}{T_a - T_s}$$

Observe that now the ratio, (ν/α), i.e. Prandtl number, enters the solution. If we draw a graph of excess temperature ratio $(T - T_s)/(T_a - T_s)$ against $(y/x).Re_x^{0.5}$ we get different curves for different Prandtl numbers; however if the excess temperature ratio is plotted against $(y/x).Re_x^{0.5}Pr^{0.333}$, we get a single curve for all Prandtl numbers and the plot is similar to that in Fig. 9.9. This plot is shown in Fig. 9.10.

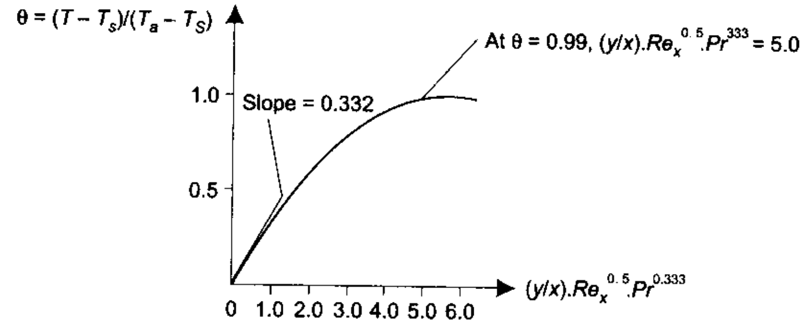

FIGURE 9.10 Dimensionless temperature ratio in laminar boundary layer, for flow over a flat plate

Again, there are two important observations to be made from Fig. 9.10:

(a) first, when the X-coordinate reaches a value of 5 , the Y-coordinate is 0.99 i.e. the local excess temperature reaches 99% of the value of total temperature difference between the free stream temperature and the plate surface temperature, when $(y/x).Re_x^{0.5}.Pr^{0.333}$ reaches a value of 5. However, from the definition of the thermal boundary layer thickness δ_t, we know that $y = \delta_t$ when $(T - T_s)/(T_a - T_s) = 99\%$. Therefore, we can write:

$$\delta_t = \frac{5 \cdot x}{\sqrt{Re_x} \cdot Pr^{0.333}} \quad \text{...(9.34)}$$

Therefore, immediately, using Eq. 9.28, we can write for the relationship between the thicknesses of hydrodynamic and thermal boundary layers:

$$\frac{\delta}{\delta_t} = Pr^{0.333} \quad \text{...(9.35)}$$

(b) second observation is that the slope at $y = 0$ is 0.332, i.e.

$$\left[\frac{d\left(\frac{T - T_s}{T_a - T_s}\right)}{d\left(\frac{y}{x} \cdot \sqrt{Re_x} \cdot Pr^{0.333}\right)}\right]_{y=0} = 0.332$$

Therefore,

$$\left(\frac{dT}{dy}\right)_{y=0} = 0.332\cdot\frac{(T_a - T_s)}{x}\cdot\sqrt{Re_x}\cdot Pr^{0.333} \quad \ldots(9.36)$$

Then, local heat transfer flux (considering the stationary layer):

$$q = -k\cdot\left(\frac{dT}{dy}\right)_{y=0} = -k\cdot 0.332\cdot\frac{(T_a - T_s)}{x}\cdot\sqrt{Re_x}\cdot Pr^{0.333} \quad \ldots(9.37)$$

Then, we have for convective heat transfer coefficient:

$$h = \frac{-k\cdot\left(\frac{dT}{dy}\right)_{y=0}}{T_s - T_a}$$

and, we can write, using Eq. 9.37:

$$\frac{h\cdot x}{k} = Nu_x = 0.332\cdot\sqrt{Re_x}\cdot Pr^{0.333} \quad \ldots(9.38)$$

And the local heat transfer coefficient is:

$$h_x = \frac{q}{T_s - T_a} = 0.332\cdot\frac{k}{x}\cdot\sqrt{Re_x}\cdot Pr^{0.333} \quad \ldots(9.39)$$

Average value of heat transfer coefficient is obtained by integrating Eq. 9.39 between $x = 0$ and $x = L$. We get:

$$h_a = 2\cdot h_{x=L} \quad \ldots(9.40)$$

i.e. average value of heat transfer coefficient is twice the local value at $x = L$.

And, then, average Nusselt number is given by:

$$Nu_a = 0.664\sqrt{Re_L}\cdot Pr^{0.333} \quad \ldots(9.41)$$

Eq. 9.41 is valid for $Pr \geq 0.6$.

In the above equations, properties of the fluid are evaluated at the mean temperature between the free stream temperature and the plate surface temperature i.e. at the 'film temperature' given by:

$$T_f = \frac{T_s + T_a}{2} \quad \ldots(9.42)$$

Eq. 9.41 is not valid for liquid metals ($Pr << 1$); for liquid metals, following correlation is suggested by Kays:

$$Nu_x = 0.565\cdot Pe_x^{0.5} \quad \ldots(Pr < .005)\ldots(9.43)$$

where $Pe_x = Re_x\cdot Pr$ = Peclet number

Example 9.2. Dry air at atmospheric pressure and 20°C is flowing with a velocity of 3 m/s along the length of a long, flat plate, 0.3 m wide, maintained at 100°C.

(a) Calculate the following quantities at $x = 0.3$ m:

(i) boundary layer thickness (ii) local friction coefficient (iii) average friction coefficient (iv) local shear stress due to friction (v) thickness of thermal boundary layer (vi) local convection heat transfer coefficient (vii) average heat transfer coefficient (viii) rate of heat transfer from the plate between $x = 0$ and $x = x$, by convection, and (ix) total drag force on the plate between $x = 0$ and $x = 0.3$ m.

(b) Also, find out the value of x_c (i.e. the distance along the length at which the flow turns turbulent, $Re_c = 5 \times 10^5$).

Solution.

Data:

$W := 0.3$ m $\quad T_s := 100°C \quad T_a := 20°C \quad U := 3$ m/s $\quad T_f := \frac{100+20}{2} = 60°C$

Properties of air are to be taken at the film temperature of 60°C. We get, from the data tables:

$\rho := 1.025$ kg/m^3 $\quad C_p := 1017$ J/(kgK) $\quad \mu := 19.907\cdot 10^{-6}$ kg/(ms) $\quad k := 0.0279$ W/(mK) $\quad Pr := 0.71$

(a) At $x = 0.3$ m:

$x := 0.3$ m

$$Re_x := \frac{\rho \cdot U \cdot x}{\mu}$$ *(local Reynolds number at $x = 0.3$ m)*

i.e. $Re_x := 4.634 \times 10^4$

Note that Re_x is less than 5×10^5. Therefore boundary layer is laminar and the equations derived above are applicable.

(i) Boundary layer thickness, δ:

We have: $$\delta := \frac{5 \cdot x}{\sqrt{Re_x}} \qquad ...(9.28)$$

i.e. $\delta = 6.968 \times 10^{-3}$ **m** ***(thickness of boundary layer)***

(ii) Local friction coefficient C_{fx}:

We have: $$C_{fx} := \frac{0.664}{\sqrt{Re_x}} \qquad ...(9.31)$$

i.e. $C_{fx} = 3.085 \times 10^{-3}$ ***(Local friction coefficient.)***

(iii) Average friction coefficient C_a:

We have: $$C_{fa} := \frac{1.328}{\sqrt{Re_x}}$$

i.e. $C_{fa} = 6.16904 \times 10^{-3}$ ***(Average friction coefficient.)***

Or: from Eq. 9.33

$$C_{fa} := 2 \cdot C_{fx} \quad \text{i.e.} \quad C_{fa} = 6.16904 \times 10^{-3}$$

(iv) Local shearing stress, τ:

We have

$$\tau := 0.332 \cdot \mu \cdot \frac{U}{x} \cdot \sqrt{Re_x} \qquad ...(9.30)$$

i.e. $\tau = 0.014$ **N/m²** ***(Local shearing stress.)***

(v) Thickness of thermal boundary layer:

We have:

$$\frac{\delta}{\delta_t} = Pr^{0.333} \qquad ...(9.35)$$

i.e. $$\delta_t := \frac{\delta}{Pr^{0.333}}$$

i.e. $\delta_t = 7.81 \times 10^{-3}$ m *(thickness of thermal boundary layer.)*

(vi) Local convection heat transfer coefficient:

We have

$$h_x := 0.332 \cdot \frac{k}{x} \cdot \sqrt{Re_x} \cdot Pr^{0.333} \qquad ...(9.39)$$

i.e. $h_x = 5.93$ **W/(m²c)** ***(local heat transfer coefficient.)***

(vii) Average heat transfer coefficient:

From Eq. 9.40, average heat transfer coefficient between $x = 0$ and $x = x$ is equal to twice the value of local heat transfer coefficient at $x = x$

i.e. $h_a := 2 \cdot h_x$

i.e. $h_a = 11.86$ **W/(m²C)** ***(average heat transfer coefficient.)***

(viii) Rate of heat transfer from the plate between $x = 0$ and $x = x$, by convection:

Area := $W \times 0.3$ m² *(area of heat transfer)*

i.e. Area = 0.09 m²

$Q := h_a \cdot \text{Area} \cdot (T_s - T_a)$

i.e. $Q = 85.395$ W *(heat transfer rate from the plate between $x = 0$ and $x = 0.3$ m.)*

(ix) Total drag force on the plate between $x = 0$ and $x = 0.3$ m

$F_D := \tau \cdot \text{Area}$, N *(drag force)*

i.e. $F_D = 1.28 \times 10^{-3}$ N.

(b) Distance at which flow turns turbulent:

We have: $Re_c = \dfrac{\rho \cdot U \cdot x_c}{\mu} = 5 \times 10^5$ *(critical Reynolds number)*

Therefore, $x_c := \dfrac{5 \cdot 10^5 \cdot \mu}{\rho \cdot U}$

i.e. $x_c = 3.237$ m ***(distance at which flow becomes turbulent.)***

Example 9.3. Dry air at atmospheric pressure and 20°C is flowing with a velocity of 3 m/s along the length of a flat plate, (size: 0.5 m × 0.25 m), maintained at 100°C. Using Blasius exact solution, calculate the heat transfer rate from: (i) the first half of the plate (ii) full plate, and (iii) next half of plate.

Solution.

Data:

$L := 0.5$ m $\quad W := 0.25$ m $\quad T_s := 100°C \quad T_a := 20°C \quad U := 3$ m/s $\quad T_f := \dfrac{100 + 20}{2} = 60°C$

Properties of air are to be taken at the film temperature of 60°C. We get, from data tables:

$\rho := 1.025$ kg/m^3 $\quad C_p := 1017$ J/(kgk) $\quad \mu := 19.907 \times 10^{-6}$ kg/(m·s) $\quad k := 0.0279$ W/(mK) $\quad Pr := 0.71$

(i) Heat transfer rate from the first half of plate:

Now, characteristic dimension to calculate Reynolds number is half the length of plate:

$x := 0.25$ m

Therefore, $Re_x := \dfrac{\rho \cdot U \cdot x}{\mu}$ *(local Reynolds no. at x = 0.5/2 = 0.25 m)*

i.e. $Re_x = 3.862 \times 10^4$

This value is less than 5×10^5; so, the boundary layer is laminar and the equations derived above are applicable:

We have:

$$h_x := 0.332 \cdot \frac{k}{x} \cdot \sqrt{Re_x} \cdot Pr^{0.333} \qquad ...(9.39)$$

i.e. $h_x = 6.496$ W/(m^{2}C) *(local heat transfer coefficient)*

Therefore average heat transfer coefficient between $x = 0$ and $x = 0.25$ m:

$h_a := 2 \cdot h_x$

i.e. $h_a = 12.992$ W/(m^{2}C) *(average heat transfer coefficient)*

Area $:= 0.25 \cdot 0.25$ m^2 *(area of half of plate)*

i.e. Area $= 0.0625$ m^2 *(area of half of plate)*

Therefore, heat transferred from first half of plate:

$Q_1 := h_a \cdot \text{Area} \cdot (T_s - T_a)$ W

i.e. $Q_1 = 64.962$ **W**

(ii) Heat transfer rate from the entire plate:

For the full plate $x = L = 0.5$ m

$L := 0.5$ m

Therefore, $Re_L := \dfrac{\rho \cdot U \cdot L}{\mu}$ *(local Reynolds no. at L = 0.5 m)*

i.e. $Re_L = 7.723 \times 10^4$

This value is less than 5×10^5; so, the boundary layer is laminar and the equations derived above are applicable:

We have:

$$h_L := 0.332 \cdot \frac{k}{L} \cdot \sqrt{Re_L} \cdot Pr^{0.333} \qquad ...(9.39)$$

i.e. $h_L = 4.594$ W/(m^{2}C) *(local heat transfer coefficient)*

Therefore average heat transfer coefficient between $x = 0$ and $x = 0.5$ m:

$h_a := 2 \cdot h_L$

i.e. $h_a = 9.187$ W/(m^{2}C) *(average heat transfer coefficient)*

Area $:= 0.5 \cdot 0.25$ m^2 *(area of full of plate)*

i.e. Area $= 0.125$ m^2 *(area of half of plate)*

Therefore, heat transferred from entire plate:

$$Q_2 := h_a \cdot \text{Area} \cdot (T_s - T_a) \text{ W}$$

i.e. $$Q_2 = 91.87 \text{ W}$$

(iii) Heat transfer rate from next half of plate:

This is equal to heat transferred from the entire plate minus the heat transferred from the first half of plate = $Q_2 - Q_1$

i.e. $$Q_2 - Q_1 = 26.908 \text{ W}$$ *(heat transferred from next half of plate)*

9.8.3 Approximate Solutions of Boundary Layer Equations–Von Karman Integral Equations

It may be observed that Blasius solution to the momentum equation, though exact, is quite cumbersome even for the simple case of a flat plate; further, much ingenuity is required in selecting a suitable similarity parameter η for the solution. In the approximate method of Von Karman, instead of developing the differential equations starting from an infinitesimal control volume, a finite control volume is selected and integral equations are developed; this may be done either by directly integrating the momentum (or energy) equation or by writing a momentum (or energy) balance for the finite control volume. This latter approach is shown below.

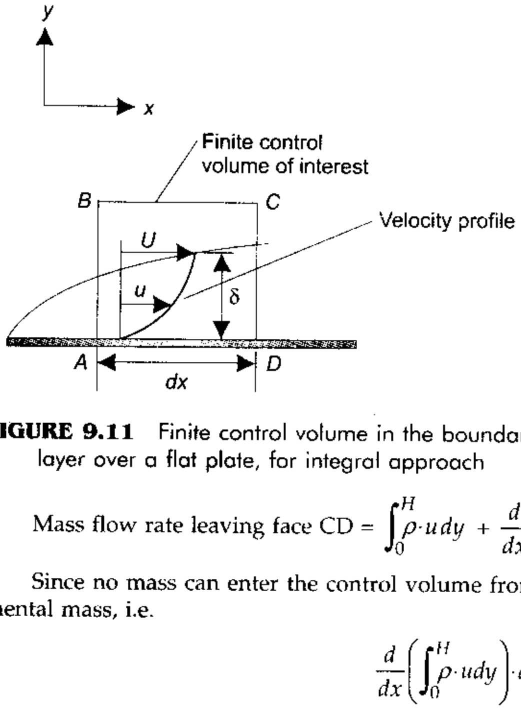

FIGURE 9.11 Finite control volume in the boundary layer over a flat plate, for integral approach

Consider a finite control volume, A-B-C-D, that extends from the wall surface in the Y-direction well into the free stream (i.e. beyond the boundary layer), has a thickness of dx in the X-direction and has unit width in the Z-direction, as shown. Let the height of AB be H (> δ). Now, let us write the momentum balance:

$$\text{Mass flow rate entering face AB} = \int_0^H \rho \cdot u\, dy$$

$$\text{Mass flow rate leaving face CD} = \int_0^H \rho \cdot u\, dy + \frac{d}{dx}\left(\int_0^H \rho \cdot u\, dy\right) \cdot dx$$

Since no mass can enter the control volume from face AD, it is clear from the mass balance that the incremental mass, i.e.

$$\frac{d}{dx}\left(\int_0^H \rho \cdot u\, dy\right) \cdot dx$$

must have entered the control volume through face BC, with the free stream velocity U.

The x-momentum fluxes are:

Influx through face AB:

$$\int_0^H \rho \cdot u^2 dy$$

efflux through face CD =

$$\int_0^H \rho \cdot u^2 dy + \frac{d}{dx}\left(\int_0^H \rho \cdot u^2 dy\right) \cdot dx$$

influx through BC =

$$U \cdot \frac{d}{dx}\left(\int_0^H \rho \cdot u\, dy\right) \cdot dx$$

Assuming that there are no pressure forces (i.e. pressure gradient in X-direction is zero) and no body forces, and also that there is no shear force on the upper face BC since it is outside the boundary layer, we write the momentum balance:

Drag or shear force at the plate surface = net momentum change for the control volume.

$$\tau_w \cdot dx = U \cdot \frac{d}{dx}\left(\int_0^H \rho \cdot u dy\right) \cdot dx - \frac{d}{dx}\left(\int_0^H \rho \cdot u^2 dy\right) \cdot dx$$

i.e.

$$\tau_w \cdot dx = U \cdot \frac{d}{dx}\left(\int_0^\delta \rho \cdot u dy\right) \cdot dx - \frac{d}{dx}\left(\int_0^\delta \rho \cdot u^2 dy\right) \cdot dx$$

Note that upper limit of integration is replaced by δ since the integrand is zero for $y > \delta$, i.e. outside the boundary layer.

Simplifying, we get:

$$\tau_w = \frac{d}{dx}\left[\int_0^\delta \rho \cdot (U - u) \cdot u dy\right]$$

i.e.

$$\tau_w = \rho \cdot U^2 \cdot \frac{d}{dx}\left[\int_0^\delta \left(1 - \frac{u}{U}\right) \cdot \frac{u}{U} dy\right] \quad ...(9.44)$$

Eq. 9.44 is known as Von Karman integral momentum equation for the boundary layer. It expresses wall shear stress τ_w as a function of non-dimensional velocity distribution (u/U). It is clear from Eq. 9.44 that if we know the velocity distribution in the boundary layer, we can calculate the wall shear stress easily.

Now, method of solution is to assume a velocity distribution in the boundary layer to start with. At first sight, this looks ridiculous to assume the velocity distribution, but since the boundary layer is very thin, assuming a velocity profile which satisfies the boundary conditions does not introduce much error. This is verified from practical results and also, as shown in Table 9.4, assumption of different velocity profiles does not give much variation in calculated values of boundary layer thickness δ or the friction coefficient C_f.

Since, from the experiments, it is observed that velocity distributions in the boundary layer at different x-locations are geometrically similar, we can say that the dimensionless velocity distribution (u/U) is a function of dimensionless distance from the wall (y/δ). So, let us assume a velocity profile as follows:

$$\frac{u}{U} = f\left(\frac{y}{\delta}\right) = a + b \cdot \left(\frac{y}{\delta}\right) + c \cdot \left(\frac{y}{\delta}\right)^2 + d \cdot \left(\frac{y}{\delta}\right)^3 \quad ...(9.45)$$

Eq. 9.45 must satisfy the following boundary conditions:

At $y = 0$: $u = 0$ and

$$\frac{d^2 u}{dy^2} = 0$$

At $y = \delta$: $u = U$ and

$$\frac{du}{dy} = 0$$

Applying these boundary conditions, we get the constants a, b, c and d in Eq. 9.45 and the velocity profile becomes:

$$\frac{u}{U} = \frac{3}{2} \cdot \left(\frac{y}{\delta}\right) - \frac{1}{2} \cdot \left(\frac{y}{\delta}\right)^3 \quad ...(9.46)$$

Let us now introduce this cubic velocity profile into the Von Karman momentum integral Eq. 9.44. Simplifying, we get:

$$\tau_w = \frac{39}{280} \cdot \rho \cdot U^2 \cdot \frac{d\delta}{dx} \quad ...(9.47)$$

At the solid surface, Newton's law of viscosity gives:

$$\tau_w = \mu \cdot \left(\frac{du}{dy}\right)_{y=0} = \mu \cdot \left[\frac{d}{dy}\left[\frac{3}{2} \cdot U \cdot \left(\frac{y}{\delta}\right) - \frac{1}{2} \cdot U \cdot \left(\frac{y}{\delta}\right)^3\right]\right]_{y=0}$$

i.e. $$\tau_w = \frac{3}{2}\cdot\frac{\mu\cdot U}{\delta} \qquad ...(9.48)$$

Equating Eqs. 9.47 and 9.48, we get:

$$\frac{39}{280}\cdot\rho\cdot U^2\cdot\frac{d\delta}{dx} = \frac{3}{2}\cdot\frac{\mu\cdot U}{\delta}$$

i.e. $$\delta\cdot d\delta = \frac{140}{13}\cdot\frac{\mu}{\rho\cdot U}\cdot dx$$

Since δ is a function of x only, we integrate the above equation and applying the condition that at $x = 0$, $\delta = 0$, we get:

$$\frac{\delta^2}{2} = \frac{140}{13}\cdot\frac{\mu\cdot x}{\rho\cdot U}$$

Or, in non-dimensional form, this may be written as:

$$\frac{\delta}{x} = \sqrt{\frac{140.2}{13}}\cdot\sqrt{\frac{\mu}{x\cdot\rho\cdot U}} = \frac{4.64}{\sqrt{Re_x}} \qquad ...(9.49)$$

where Re_x is the Reynolds number with characteristic distance x from the leading edge of the plate.

Eq. 9.49 gives the boundary layer thickness δ, at a distance x from the leading edge.

To calculate the shear stress at the wall, let us insert this value of δ in the expression 9.48 for τ_w:

$$\tau_w = \frac{3}{2}\cdot\frac{\mu\cdot U}{\delta} = \frac{3}{2}\cdot\frac{\mu\cdot U}{\frac{4.64\cdot x}{\sqrt{Re_x}}}$$

i.e $$\tau_w = \frac{\rho\cdot U^2}{2}\cdot\frac{0.646}{\sqrt{Re_x}}$$

Now, from the definition of local skin friction coefficient, we have:

$$C_{fx} = \frac{\tau_w}{\frac{1}{2}\cdot\rho\cdot U^2} = \frac{0.646}{\sqrt{Re_x}} \qquad ...(9.50)$$

Average skin friction coefficient is given by:

$$C_{fa} = \frac{1}{L}\cdot\int_0^L C_{fx}\;dx = \frac{1}{L}\cdot\int_0^L \frac{0.646}{\sqrt{\frac{\rho\cdot U}{\mu}}\cdot\sqrt{x}}dx$$

i.e. $$C_{fa} = 1.292\cdot\sqrt{\frac{\mu}{L\cdot\rho\cdot U}} = \frac{1.292}{\sqrt{Re_L}} \qquad ...(9.51)$$

where Re_L is the Reynolds number based on length L of the plate.

Note that values of boundary layer thickness and skin friction coefficient obtained above with the approximate, integral method, match reasonably well with the values obtained by exact analysis of Blasius.

Further, if we assume a velocity profile other than the cubic velocity profile assumed above (satisfying the boundary conditions), it is observed that the results obtained do not differ greatly. Table 9.4 demonstrates this fact for some velocity profiles, including linear, parabolic and cubic. Blasius exact results are shown for comparison:

Note that above results are valid for laminar boundary layer conditions only.

Mass flow through the boundary:

If we consider a section at any distance x from the leading edge, mass flow through that section is given by: $m_x = \int$ [Area × Velocity × density]; integration is performed within the limits 0 to δ.

Table 9.4 Boundary layer thickness (δ) and skin friction coefficient (C_{fa}) for different velocity profiles

Velocity profile	Boundary conditions		$\frac{\delta}{x}\cdot\sqrt{Re_x}$	$C_{fa}\cdot\sqrt{Re_L}$
	At $y = 0$	At $y = \delta$		
1. $\frac{u}{U} = \frac{y}{\delta}$	$u = 0$	$u = U$	3.46	1.155
2. $\frac{u}{U} = 2\cdot\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$	$u = 0$	$u = U$	5.47	1.462
		$\frac{du}{dy} = 0$		
3. $\frac{u}{U} = \frac{3}{2}\cdot\frac{y}{\delta} - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^3$	$u = 0$	$u = U$	4.64	1.292
	$\frac{d^2u}{dy^2} = 0$	$\frac{du}{dy} = 0$		
4. $\frac{u}{U} = \sin\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right)$	$u = 0$	$u = U$	4.78	1.310
5. Blasius exact solution			5.0	1.328

i.e.

$$m_x = \int_0^{\delta} \rho\cdot u dy$$

Assuming the cubic velocity profile as done earlier, substituting for u, we get:

$$m_x = \int_0^{\delta} \rho\cdot\left[U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^3\right]\right]dy$$

$$m_x = \rho\cdot U\cdot\left[\frac{3}{4}\times\frac{y^2}{\delta} - \frac{1}{8}\times\frac{y^4}{\delta^3}\right]_0^{\delta}$$

i.e.

$$m_x = \frac{5}{8}\cdot\rho\cdot U\cdot\delta \qquad ...(9.52)$$

Mass entrained between two sections at x_1 and x_2 can be calculated using Eq. 9.52 as:

$$\delta m = \frac{5}{8}\cdot\rho\cdot U\cdot(\delta_2 - \delta_1) \qquad ...(9.53)$$

where δ_1 and δ_2 are the thicknesses of boundary layer at sections x_1 and x_2 respectively.

Integral energy equation:

Von Karman integral technique may be applied to get an approximate solution for the energy equation of the boundary layer, as shown below:

Consider a finite control volume that encloses both the hydrodynamic and thermal boundary layers, (laminar and incompressible) as shown in Fig. 9.12. Assume that the fluid properties do not vary with temperature and are constant; let the heating of the plate commence at a distance x_0 from the leading edge of the plate. That means that thermal boundary layer develops only beyond x_0 from the leading edge.

Le us make an energy balance on the control volume ABCD.

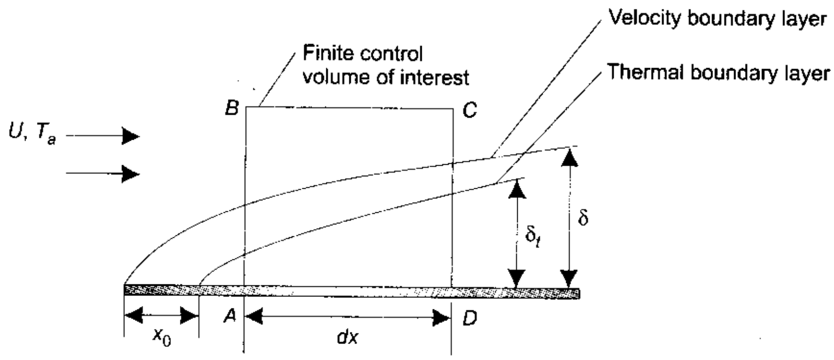

FIGURE 9.12 Finite control volume in the boundary layer over a flat plate, for integral energy equation

Energy enters the control volume by convection at face AB, leaves by convection at face CD; also, energy enters the control volume by conduction through face AD and by convection through face BC. Let us write the various terms involved:

Fluid mass entering face – AB = $\int_0^H \rho \cdot u dy$

Fluid mass leaving through face – CD = $\int_0^H \rho \cdot u dy + \frac{d}{dx}\left(\int_0^H \rho \cdot u dy\right) \cdot dx$

From continuity consideration, mass increment viz.

$$\frac{d}{dx}\left(\int_0^H \rho \cdot u dy\right) \cdot dx$$

must enter the control volume from top face BC.

Heat fluxes through the four faces are:

Heat influx through AB = Q_x = mass X specific heat X temperature

i.e.

$$Q_x = \int_0^H \rho \cdot u dy \cdot C_p \cdot T \quad \text{...(a)}$$

Heat efflux through CD =

$$Q_{x+dx} = \int_0^H \rho \cdot u dy \cdot C_p \cdot T + \frac{d}{dx}\left(\int_0^H \rho \cdot u dy \cdot C_p \cdot T\right) \cdot dx \quad \text{...(b)}$$

Heat influx through upper face BC: Since face BC is well outside the thermal boundary layer, its temperature is equal to free stream temperature, T_a. So, we have:

$$Q_{BC} = \frac{d}{dx}\left(\int_0^H \rho \cdot u dy\right) \cdot C_p \cdot T_a \quad \text{...(C)}$$

Heat conducted into the control volume through lower face AD =

$$Q_{AD} = -k \cdot A \cdot \left(\frac{d\tau}{dy}\right)_{y=0}$$

i.e

$$Q_{AD} = -k \cdot dx \cdot \left(\frac{dT}{dy}\right)_{y=0} \quad \text{...(d)}$$

Writing the energy balance:

Heat flow into the control volume = Heat flow out of the control volume or,

$$Q_x + Q_{BC} + Q_{AD} = Q_{x+dx}$$

Substituting from Eqs. a, b, c and d and simplifying, we get:

$$\frac{d}{dx}\left[\int_0^H (T_a - T)\cdot u dy\right] = \frac{k}{\rho \cdot C_p}\cdot\left(\frac{dT}{dy}\right)_{y=0}$$

i.e.

$$\frac{d}{dx}\left[\int_0^H (T_a - T)\cdot u dy\right] = \alpha\cdot\left(\frac{dT}{dy}\right)_{y=0} \quad ...(9.54)$$

Eq. 9.54 is the integral energy equation for the boundary layer, with constant thermo-physical properties and constant free stream temperature.

Note that we have neglected viscous dissipation in the element since it is very small for low velocities.

To solve the integral energy equation we have to assume the velocity and temperature profiles; let us assume cubic velocity profile and cubic temperature profiles.

Cubic velocity profile, as shown earlier, is:

$$\frac{u}{U} = \frac{3}{2}\cdot\frac{y}{\delta} - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^3 \quad ...(9.55)$$

Temperature distribution must satisfy the boundary conditions:

At $y = 0$: $T = T_s$ and $\frac{d^2T}{dy^2} = 0$

At $y = \delta_t$: $T = T_a$ and $\frac{dT}{dy} = 0$

These boundary conditions are of the form as required for the velocity profile; therefore, temperature distribution is also of the form:

$$\frac{\theta}{\theta_a} = \frac{T - T_s}{T_a - T_s} = \frac{3}{2}\cdot\left(\frac{y}{\delta_t}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta_t}\right)^3 \quad ...(9.56)$$

where T_s is the plate surface temperature, T_a is the free stream temperature and δ_t is the thickness of thermal boundary layer at a given section.

Now, the Eqs. 9.55 and 9.56 are inserted in the integral Eq. 9.54 and simplified. For most gases ($Pr \cong 1$) and oils ($Pr > 1$), thermal boundary layer is thinner than hydrodynamic boundary layer, i.e. $\delta_t < \delta$; so, upper limit of integration is changed to δ_t instead of H because the integrand becomes zero beyond $y = \delta_t$.

Final solution for the thermal boundary layer thickness is:

$$\frac{\delta_t}{\delta} = \frac{0.976}{Pr^{\frac{1}{3}}}\cdot\left[1 - \left(\frac{x_0}{x}\right)^{\frac{3}{4}}\right]^{\frac{1}{3}} \quad ...(9.57)$$

Remember that Eq. 9.57 is for the case when the heating of the plate starts at a distance of x_0 from the leading edge. Instead, if the heating starts from the leading edge itself, putting $x_0 = 0$, we get:

$$\frac{\delta_t}{\delta} = \frac{0.976}{Pr^{\frac{1}{3}}} \quad ...(9.58)$$

Observe that this value of δ_t is close to the value obtained with exact analysis.

Local heat transfer coefficient (h_x):

We obtain h_x from the relation:

$$h_x = \frac{-k \cdot \left(\frac{dT}{dy}\right)_{y=0}}{T_s - T_a}$$

Getting dT/dy from Eq. 9.56, and taking the values of δ and (δ_t/δ) from Eqs. 9.49 and 9.57 respectively, we get:

$$h_x = 0.332 \cdot \frac{k}{x} \cdot Re_x^{\frac{1}{2}} \cdot Pr^{\frac{1}{3}} \cdot \frac{1}{\left[1-\left(\frac{x_0}{x}\right)^{\frac{3}{4}}\right]^{\frac{1}{3}}} \qquad ...(9.59)$$

and, in terms of non-dimensional Nusselt number, we write:

$$Nu_x = \frac{h_x \cdot x}{k} = \frac{0.332 \cdot Re_x^{\frac{1}{2}} \cdot Pr^{\frac{1}{3}}}{\left[1-\left(\frac{x_0}{x}\right)^{\frac{3}{4}}\right]^{\frac{1}{3}}} \qquad ...(9.60)$$

If the plate is heated over the entire length, $x_0 = 0$, and we get:

$$h_x = 0.332 \cdot \frac{k}{x} \cdot Re_x^{\frac{1}{2}} \cdot Pr^{\frac{1}{3}} \qquad ...(9.61)$$

and,

$$Nu_x = \frac{h_x \cdot x}{k} = 0.332 \cdot Re_x^{\frac{1}{2}} \cdot Pr^{\frac{1}{3}} \qquad ...(9.62)$$

Note that Eq. 9.62 is in excellent agreement with the value obtained with exact analysis.

Average value of the heat transfer coefficient is obtained by integrating the local value over the entire plate:

$$h_a = \frac{1}{L} \cdot \int_0^L h_x \, dx$$

Performing the integration after substituting for h_x, we get:

$$h_a = 2 \cdot h_L \qquad ...(9.63)$$

Similarly, average value of Nusselt number is obtained as:

$$Nu_a = \frac{h_a \cdot L}{k} = 0.664 \cdot Re_L^{\frac{1}{2}} \cdot Pr^{\frac{1}{3}} \qquad ...(9.64a)$$

or,

$$Nu_a = 2 \cdot Nu_L \qquad ...(9.64b)$$

where

$$Re_L = \frac{\rho \cdot U \cdot L}{\mu}$$

Note that all the above analysis is for laminar boundary layer conditions; property values are taken at film temperature (i.e. mean value of surface and free stream temperatures), given by:

$$T_f = \frac{T_s + T_a}{2}$$

Eq. 9.61 is valid for fluids with Prandtl numbers varying from 0.6 to 50 i.e. it is not applicable to liquid metals for whom $Pr << 0.6$ and for heavy oils or silicones for whom $Pr >> 50$.

For a wide range of Prandtl numbers Churchill and Ozoe have given the following correlation, for laminar flow on an **isothermal flat plate:**

$$Nu_x = \frac{0.3387 \cdot Re_x^{\frac{1}{2}} \cdot Pr^{\frac{1}{3}}}{\left[1+\left(\frac{0.0468}{Pr}\right)^{\frac{2}{3}}\right]^{\frac{1}{4}}}$$ ...*for* $Re_x\, Pr > 100$ ...(9.65)

For constant heat flux conditions:

All the above relations were derived for laminar flow over a flat plate, temperature of the plate being maintained constant. However, there are many practical cases where the heat flux over the surface is constant (e.g. when the surface is heated by electrical heaters).

For the case of constant heat flux, it is shown that local Nusselt number is given by:

$$Nu_x = \frac{h \cdot x}{k} = 0.453 \cdot Re_x^{\frac{1}{2}} \cdot Pr^{\frac{1}{3}}$$ ...$Pr \geq 0.6$...(9.66)

In terms of surface heat flux and temperature difference, this is written as:

$$Nu_x = \frac{q_s \cdot x}{k \cdot (T_s - T_a)}$$ ...(9.67)

Average temperature difference along the plate for this case is obtained by performing the integration:

$$(T_s - T_a)_{avg} = \frac{1}{L} \cdot \int_0^L (T_s - T_a)\,dx$$

Substituting for $(T_s - T_a)$ from Eq. 9.67 and performing the integration, we get:

$$(T_s - T_a)_{avg} = \frac{q_s \cdot \frac{L}{k}}{0.6795\, Re_L^{\frac{1}{2}} \cdot Pr^{\frac{1}{3}}}$$ ...(9.68)

and, $$q_s = \frac{3}{2} \cdot h_L \cdot (T_s - T_a)_{avg}$$ ...(9.69)

In the above equations, q_s is the heat flux per unit area with units: W/m^2.

Again, for the constant heat flux case, Eq. 9.65 for very wide range of Prandtl numbers, is modified as:

$$Nu_x = \frac{0.4637 \cdot Re_x^{\frac{1}{2}} \cdot Pr^{\frac{1}{3}}}{\left[1+\left(\frac{0.02052}{Pr}\right)^{\frac{2}{3}}\right]^{\frac{1}{4}}}$$ ...*for* $Re_x\, Pr > 100$ ...(9.70)

Fluid properties are still evaluated at the film temperature.

In all cases, average Nusselt number is $Nu_a = 2 \cdot Nu_L$ ... (9.70a)

Example 9.4. Air at 20°C and atmospheric pressure is flowing with a velocity of 3 m/s along the length of a flat plate, maintained at 60°C. Calculate: (i)hydrodynamic boundary layer thickness at 20 cm and 40 cm from the leading edge, by the approximate method (ii) mass entrainment rate between these two sections assuming a cubic velocity profile, and (iii) heat transferred from the first 40 cm of the plate.

Solution.

Data:

$T_s := 60°C$ $T_a := 20°C$ $U := 3.0$ m/s $x_1 := 0.2$ m $x_2 := 0.4$ m $T_f = \frac{60 + 20}{2} = 40°C$

Properties of air are to be taken at the film temperature of 40°C. We get, from data tables:

$\rho := 1.092$ kg/m³ $C_p := 1014$ J/(kgK) $\mu := 19.123 \times 10^{-6}$ NS/m² $k := 0.0265$ W/(mK) $Pr := 1.01$

(i) Hydrodynamic boundary layer thickness at section-1 (i.e. $x = 0.2$ m) of plate:
Now, characteristic dimension to calculate Reynolds number is length x_1

Therefore, $Re_{x1} := \frac{\rho \cdot U \cdot x_1}{\mu}$ *(local Reynolds no. at $x_1 = 0.2$ m)*

i.e. $Re_{x1} = 3.426 \times 10^4$

This value is less than 5×10^5; so, the boundary layer is laminar and the equations derived above are applicable:
Hydrodynamic boundary layer thickness, δ_1:

We have: $\delta_1 := \frac{4.64 \cdot x_1}{\sqrt{Re_{x1}}}$ ...(9.49)

i.e. $\delta_1 = 5.013 \times 10^{-3}$ **m** ***(thickness of hydrodynamic boundary layer at $x_1 = 0.2$ m.)***

Hydrodynamic boundary layer thickness at section-2 (i.e. $x = 0.4$ m) of plate:
Now, characteristic dimension to calculate Reynolds number is length x_2:

Therefore, $Re_{x2} := \frac{\rho \cdot U \cdot x_2}{\mu}$ *(local Reynolds no. at $x_2 = 0.4$ m)*

i.e. $Re_{x2} = 6.852 \times 10^4$

This value is less than 5×10^5; so, the boundary layer is laminar and the equations derived above are applicable:
Hydrodynamic boundary layer thickness δ_2:

We have: $\delta_2 := \frac{4.64 \cdot x_2}{\sqrt{Re_{x2}}}$ ...(9.49)

i.e. $\delta_2 = 7.09 \times 10^{-3}$ **m** ***(thickness of hydrodynamic boundary layer at $x_2 = 0.2$ m.)***

(ii) Mass flow entrained between sections 1 and 2:
For a cubic velocity profile, mass flow entrained between section 1 and 2 is already shown to be:

$$\delta m := \frac{5}{8} \cdot \rho \cdot U \cdot (\delta_2 - \delta_1) \qquad ...(9.53)$$

i.e. $\delta m = 4.252 \times 10^{-3}$ **kg/s** ***(mass entrained between sections 1 and 2.)***

(iii) heat transferred from the first 40 cm of the plate:
Now, we have: $Re_{x2} = 6.852 \times 10^4$

$$Nu_x := 0.332 \cdot Re_{x2}^{\frac{1}{2}} \cdot Pr^{\frac{1}{3}} \qquad ...(9.62)$$

i.e. $Nu_x = 87.197$ *(Nusselt number)*

But, $Nu_x = \frac{h_x \cdot x}{k}$

Therefore, $h_x := \frac{Nu_x \cdot k}{x_2}$

i.e. $h_x = 5.777$ W/(m^{2}C) *(local heat transfer coefficient)*

To calculate the heat transferred from first 40 cm of the plate, we need the average value of heat transfer coefficient over this length. It is given by twice the value of local heat transfer coefficient at $x = 0.4$ m. i.e.

$h_a := 2 \cdot h_x$ ...(9.63)

i.e. $h_a = 11.554$ W/(m^{2}C) *(average heat transfer coefficient over 40 cm length)*

Area := 0.4.1 m^2 *(heat transfer area for unit width)*

Therefore, heat transferred over 40 cm length of plate:

$Q := h_a \cdot \text{Area} \cdot (T_s - T_a)$ W

i.e. $Q = 184.858$ W ***...heat transferred over 40 cm length of plate.***

Example 9.5. Engine oil at 30°C is flowing with a velocity of 2 m/s along the length of a flat plate, maintained at 90°C. Calculate, at a distance of 40 cm from the leading edge: (i) hydrodynamic and thermal boundary layer thicknesses by the exact method (ii) local and average values of friction coefficient (iii) local and average values of heat transfer coefficient, and (iv) heat transferred from the first 40 cm of the plate for unit width.

Solution.

Data:

$T_s := 90°C \quad T_a := 30°C \quad U := 2.0$ m/s $\quad x := 0.4$ m $\quad T_f = \frac{90 + 30}{2} = 60°C$

Properties of engine oil are to be taken at the film temperature of 60°C. We get, from data tables:

$\rho := 864\ \text{kg/m}^3 \quad C_p := 2047\ \text{J/(kgK)} \quad \mu := 72.5 \times 10^{-3}\ \text{Ns/m}^2 \quad k := 0.140\ \text{W/(mK)} \quad Pr := 1050$

(i) Hydrodynamic and thermal boundary layer thickness at 0.4 m from leading edge of plate:

Now, characteristic dimension to calculate Reynolds number is: $x = 0.4$ m

Therefore, $$Re_x := \frac{\rho \cdot U \cdot x}{\mu}$$ *(local Reynolds no. at $x = 0.4$ m)*

i.e. $$Re_x = 9.534 \times 10^3$$

This value is less than 5×10^5; so, the boundary layer is laminar and the equations derived earlier are applicable:

Hydrodynamic boundary layer thickness δ:

We have: $$\delta := \frac{5 \cdot x}{\sqrt{Re_x}} \quad \text{...(9.28)}$$

i.e. $$\delta = 0.02\ \text{m}$$ ***(thickness of hydrodynamic boundary layer.)***

Thickness of thermal boundary layer:

We have:

$$\frac{\delta}{\delta_t} = Pr^{0.333} \quad \text{...(9.35)}$$

i.e. $$\delta_t := \frac{\delta}{Pr^{0.333}}$$

i.e. $$\delta_t = 2.02 \times 10^{-3}\text{m}$$ ***(thickness of thermal boundary layer.)***

Note that thermal boundary layer thickness is very small compared to that of hydrodynamic boundary layer, since $Pr >> 1$.

(ii) Local and average values of friction coefficient:

We have:

$$C_{fx} := \frac{0.664}{\sqrt{Re_x}} \quad \text{...(9.31)}$$

i.e. $$C_{fx} = 6.8 \times 10^{-3}$$ ***(value of local friction coefficient.)***

And,

$$C_{fa} := 2 \cdot Cf_x \quad \text{...(9.33)}$$

i.e. $$C_{fa} = 0.014$$ ***(value of average friction coefficient)***

(iii) Local and average values of heat transfer coefficient:

Since Prandtl number is very high and $Re_x \cdot Pr = 1.001 \times 10^7 > 100$,

we shall use Eq. (9.65), i.e.

$$Nu_x := \frac{0.3387 \cdot Re_x^{\frac{1}{2}} \cdot Pr^{\frac{1}{3}}}{\left[1 + \left(\frac{0.0468}{Pr}\right)^{\frac{2}{3}}\right]^{\frac{1}{4}}}$$ *(for $Re_x\, Pr > 100$...(9.65))*

i.e. $$Nu_x = 336.027$$ *(Nusselt number)*

And, $$h_x := \frac{Nu_x \cdot k}{x}$$ *(local value of heat transfer coefficient)*

i.e. $$h_x = 117.61\ \text{W/(m}^2\text{C)}$$ ***(value of local heat transfer coefficient.)***

Therefore, average value of heat transfer coefficient

$$h_a := 2 \cdot h_x$$

i.e. $$h_a = 235.219\ \text{W/m}^2\text{C)}$$ ***(value of average heat transfer coefficient.)***

(iv) Heat transferred from the first 40 cm of the plate for unit width.

$$\text{Area} := 0.4\ \text{m}^2$$ *(area of heat transfer for unit width)*

Therefore, $$Q := h_a\ \text{Area} \cdot (T_s - T_a)$$

i.e. $$Q = 5.645 \times 10^3\ \text{W}$$ ***(heat transfer rate from the plate between $x = 0$ and $x = 0.4$ m***

Example 9.6. An air stream at 20°C and atmospheric pressure, flows with a velocity of 5 m/s over an electriacally heated flat plate (size: 0.5 m × 0.5 m), heater power being 1 kW. Calculate:
(i) the average temperature difference along the plate (ii) heat transfer coefficient, and (iii) temperature of the plate at the trailing edge

Solution.

Data:

$$T_a := 20°C \qquad U := 5.0 \text{ m/s} \qquad L := 0.5 \text{ m} \qquad W := 0.5 \text{ m} \qquad q_s := \frac{1000}{0.5 \cdot 0.5} \quad \text{i.e.} \quad q_s = 4 \times 10^3 \text{ W/m}^2$$

Note that properties have to be evaluated at the film temperature; however, since the temperature of the plate is not constant, but varies along the length, we shall start the analysis taking the properties at 20°C and then refine the values later.

At 20°C and atmospheric pressure, properties of air are:

$\rho := 1.205$ kg/m^3 $\quad C_p := 1005$ J/(kgK) $\quad \nu := 15.06 \times 10^{-6}$ m^2/s $\quad k := 0.02593$ W/(mK) $\quad Pr := 0.703$

(i) the average temperature difference along the plate

First check Reynolds number for laminar flow:

$$Re_L := \frac{U \cdot L}{\nu} \quad \text{i.e.} \quad Re_L = 1.66 \times 10^5 < 5 \times 10^5$$

Therefore, flow is laminar.

For constant heat flux conditions, we use Eq. 9.68, to calculate the average temperature difference:

$$(T_s - T_a)_{avg} = \frac{q_s \cdot \frac{L}{k}}{0.6795 \cdot Re_L^{\frac{1}{2}} \cdot Pr^{\frac{1}{3}}} \qquad \text{...(9.68)}$$

i.e. $(T_s - T_a)_{avg} = 313.325$ deg. C

Now, find the properties again at a film temperature of: (20 + 313.3)/2 = 161.5°C

We get:

$\nu := 30.1 \times 10^{-6}$ m^2/s $\quad k := 0.0365$ W/(mK) $\quad Pr := 0.682$

Now, using Eq. 9.68 again, we get:

$$Re_L := \frac{U \cdot L}{\nu} \quad \text{i.e.} \quad Re_L = 8.306 \times 10^4$$

and, $(T_s - T_a)_{avg} = 317.882$ deg.C

Therefore, film temperature:

$$\frac{317.88 + 20}{2} = 168.94°C$$

Now, again, find the properties again at a film temperature of: 169°C

We get:

$\nu := 31.25 \times 10^{-6}$ m^2/s $\quad k := 0.0371$ W/(mK) $\quad Pr := 0.6815$

Now, using Eq. 9.68 again, we get:

$$Re_L := \frac{U \cdot L}{\nu} \quad \text{i.e.} \quad Re_L = 8 \times 10^4$$

and, $(T_s - T_a)_{avg} = 318.737$ deg. C

Therefore, film temperature:

$$\frac{318.737 + 20}{2} = 169.369°C$$

Therefore, the film, temperature has not changed much. So, we conclude:

$(T_s - T_a)_{avg} = 318.737$ **deg. C** *(Average value of temperature difference over the plate length.)*

(ii) Convection heat transfer coefficient:

We have, for the case of constant heat flux:

$$Nu_x = \frac{h \cdot x}{k} = 0.453 \cdot Re_x^{\frac{1}{2}} \cdot Pr^{\frac{1}{3}} \qquad \text{...(9.66)}$$

Now,

$$Re_L := \frac{U \cdot L}{\nu} \quad \text{i.e.} \quad Re_L = 8 \times 10^4$$ *(Note: taking v at 169°C)*

From Eq. 9.66: $$Nu_L := 0.453 \cdot Re_L^{\frac{1}{2}} \cdot Pr^{\frac{1}{3}}$$

i.e. $$Nu_L = 112.754$$

Therefore, $$h_L := \frac{Nu_L \cdot k}{L}$$

i.e. $$h_L = 8.366 \text{ W/(m}^2\text{C)}$$ *(local heat transfer coefficient at the end of plate, i.e. at x = L)*

Therefore, average heat transfer coefficient over the whole length of plate:

$$h_{avg} := 2 \cdot h_L$$

i.e. $$h_{avg} = 16.733 \text{ W/(m}^2\text{C)}$$ ***(average heat transfer coefficient over the plate.)***

(iii) Temperature of the plate at the trailing edge:

From the basic heat flow equation., we have:

$$T_s - T_a = \frac{q_s \cdot L}{Nu_L \cdot k} \quad \text{as} \quad h = \frac{Nu_L \cdot k}{L}$$

i.e. • $$T_s - T_a = 478.1$$

i.e. $$T_s = 498.1°\text{C}$$ ***...temperature of plate at trailing edge.***

Example 9.7. Sodium-Potassium alloy (25% + 75 %), at 300°C , flows with a velocity of 0.4 m/s over a flat plate (size: 0.3 m × 0.1 m), maintained at 500°C. Calculate (i) the hydrodynamic and thermal boundary layer thicknesses (ii) local and average value of friction coefficient (iii) heat transfer coefficient, and (iv) total heat transfer rate

Solution.

Data:

$$T_s := 500°\text{C} \quad T_a := 300°\text{C} \quad U := 0.4 \text{ m/s} \quad L := 0.3 \text{ m} \quad W := 0.1 \text{ m} \quad T_f = \frac{300 + 500}{2} = 400°\text{C}$$

Properties of Na–K alloy are to be taken at the film temperature of 400°C. We get, from data tables:

$$\nu := 0.308 \times 10^{-6} \text{ m}^2/\text{s} \quad k := 22.10 \text{ W/(mK)} \quad Pr := 0.0108$$

(i) Hydrodynamic and thermal boundary layer thickness at 0.3 m from leading edge of plate:

Now, characteristic dimension to calculate Reynolds number is: $x = 0.3$ m

Therefore, $$Re_L := \frac{U \cdot L}{\nu}$$ *(local Reynolds no. at L = 0.3 m)*

i.e. $$Re_L = 3.896 \times 10^5$$

This value is less than 5×10^5; so, the boundary layer is laminar and the equations derived earlier are applicable:

Hydrodynamic boundary layer thickness, δ:

We have: $$\delta := \frac{5 \cdot L}{\sqrt{Re_L}} \qquad ...(9.28)$$

i.e. $$\delta = 2.403 \times 10^{-3} \text{ m}$$ ***(thickness of hydrodynamic boundary layer.)***

Thickness of thermal boundary layer, δ_t:

We have:

$$\frac{\delta}{\delta_t} = Pr^{0.333} \qquad ...(9.35)$$

i.e. $$\delta_t := \frac{\delta}{Pr^{0.333}}$$

i.e. $$\delta_t := 0.011 \text{ m}$$ ***(thickness of thermal boundary layer.)***

Note that thermal boundary layer thickness is very large compared to that of hydrodynamic boundary layer, since $Pr << 1$.

(ii) Local and average values of friction coefficient:

We have:

$$C_{fL} := \frac{0.664}{\sqrt{Re_L}} \qquad ...(9.31)$$

i.e. $C_{fL} = 1.064 \times 10^{-3}$ *(local value of friction coefficient)*

And,

$$C_{fa} := 2 \cdot C_{fL} \quad \text{...(9.33)}$$

i.e. $C_{fa} = 2.128 \times 10^{-3}$ ***(average value of friction coefficient)***

(iii) Local and average values of heat transfer coefficient:

Since Prandtl number is very low (liquid metal) and $Re_L \cdot Pr = 4.208 \times 10^3 > 100$, we shall use Eq. 9.65 i.e.

$$Nu_x := \frac{0.3387 \cdot Re_x^{\frac{1}{2}} \cdot Pr^{\frac{1}{3}}}{\left[1 + \left(\frac{0.0468}{Pr}\right)^{\frac{2}{3}}\right]^{\frac{1}{4}}} \quad \text{(for } Re_x\, Pr > 100\text{...(9.65))}$$

i.e. $Nu_L = 33.791$ *(Nusselt number)*

And, $h_L := \frac{Nu_L \cdot k}{L}$ *(local value of heat transfer coefficient)*

i.e. $h_L = 2.489 \times 10^3$ **W/(m²C)** ***(local value of heat transfer coefficient)***

Therefore, average value of heat transfer coefficient

$$h_a := 2 \cdot h_L$$

i.e. $h_a = 4.978 \times 10^3$ **W/(m²C)** ***(average value of heat transfer coefficient)***

(iv) Heat transferred from the plate:

Area $:= 0.03\ \text{m}^2$ *(area of heat transfer)*

Therefore, $Q := h_a \cdot \text{Area} \cdot (T_s - T_a)$

i.e. $Q = 2.987 \times 10^4$ **W** ***(heat transfer rate from the plate between x = 0 and x = 0.4 m)***

Note: Alternatively, for liquid metals, we can also use Eq. 9.43 to get local Nusselt number:

$$Nu_x := 0.565\ Pe_x^{0.5} \quad ((Pr < 005)\text{...(9.43))}$$

where Pe is the Peclet number = $Re.Pr$

i.e. $Nu_L := 0.565 \cdot (Re_L \cdot Pr)^{0.5}$

i.e. $Nu_L = 36.65$

Compare this value of Nusselt number with the value of 33.791, obtained from Eq. 9.65.

Then, $h_L := \frac{Nu_L \cdot k}{L}$ *(local value of heat transfer coefficient)*

i.e. $h_L = 2.7 \times 10^3$ **W/(m²C)** ***(local value of heat transfer coefficient)***

Therefore, average value of heat transfer coefficient

$$h_a := 2 \cdot h_L$$

i.e. $h_a = 5.4 \times 10^3$ **W/(m²c)** ***(average value of heat transfer coefficient)***

And, $Q := h_a \cdot \text{Area} \cdot (T_s - T_a)$

i.e. $Q = 3.24 \times 10^4$ **W** ***(heat transfer rate from the plate between x = 0 and x = 0.4 m.)***

Value of Q thus obtained is about 8.5% higher than the value obtained by using Eq. 9.65.

9.2.3.1 Turbulent boundary layer flow over a flat plate. Consider a flat plate over which a fluid flows with a free stream velocity of U. At the leading edge the fluid comes in contact with the surface and then along the length a boundary layer develops, as explained earlier. For a certain distance from the leading edge the flow in the boundary layer is 'laminar', i.e. the flow is regular and the layers of fluid are all parallel to each other; however, after this distance, called 'critical distance' (x_c), the flow becomes 'turbulent', i.e. the flow becomes highly irregular and there is completely random motion of fluid chunks. The transition from laminar to turbulent is not sudden, but there is a transition region in between. The dimensionless number characterizing the type of flow i.e. whether it is laminar or turbulent, is the Reynolds number, Re (= $\rho.U.L/\mu$). For a flat plate, generally accepted value of Re at which flow becomes turbulent is 5×10^5; however, it should be understood that this value is not a fixed value, but depends on the surface conditions i.e. if the surface is smooth or rough.

The turbulent boundary layer itself is thought of as subdivided into three sections viz. a laminar sub-layer, a buffer layer and lastly, a turbulent region. See Fig. 9.2.

Now, one could easily imagine that because of the nature of random motion of fluid in turbulent flow, an exact mathematical analysis of this phenomenon is rather difficult. Models have been proposed by many research workers to explain the observed phenomenon: Reynolds conducted his famous 'dye experiment' in 1883 to visually demonstrate the transition from laminar to turbulent flow. In turbulent flow it is observed that secondary motions of the fluid are superimposed on the main flow and there are irregular fluctuations of local velocity. Chunks of fluid, called 'eddies' move across the line of motion causing mixing of the fluid, thus causing the transport of momentum as well as energy. Therefore, in turbulent flow, heat transfer is enhanced; also, there is increased 'drag force' or pressure drop. Prandtl (1925) suggested that the 'eddies' moving across the fluid layers cause the transport of momentum, and the average transverse distance moved by an eddy before it gets mixed with other particles and loses its identity is called as 'mixing length'. This mixing length is akin to the 'mean free path' appearing in the kinetic theory of gases.

Turbulent flow is important in heat transfer applications, since there is increased heat transfer in turbulent flow; of course, this is achieved with a penalty of increased pressure drop. It is usual to introduce 'turbulence promoters' in applications where increased heat transfer is the primary consideration.

We shall not go into the theories of turbulence, but give here the important results useful for practical applications.

Velocity distribution:

Boundary layer thickness is more in turbulent flow as compared to that in laminar flow. Also, the velocity distribution is more uniform across the thickness of boundary layer as shown in Fig. 9.2. It is observed experimentally that the velocity distribution in turbulent flow follows the one-seventh power law:

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \quad ...(9.71)$$

Surface shear stress:

Surface shear stress is given by:

$$\tau_w = 0.0225\rho \cdot U^{\frac{7}{4}} \cdot \left(\frac{y}{\delta}\right)^{\frac{1}{4}} \quad ...(9.72)$$

Hydrodynamic boundary layer thickness:

This is obtained by solving the integral momentum equation, i.e.

$$\tau_w = \frac{d}{dx}\left[\int_0^{\delta} \rho \cdot (U-u) \cdot u dy\right]$$

Substituting for $u(y)$ and τ_w from Eqs. 9.71 and 9.72 respectively, and solving, we get:

$$\frac{\delta}{x} = 0.371 \cdot Re_x^{\frac{-1}{5}} \quad ...(9.73)$$

Thermal boundary layer thickness:

In turbulent flow, since the effects of physical movement of eddies predominates over the diffusion effects, Prandtl number does not have much influence on the thermal boundary layer thickness, δ_t and is of the same order as the hydrodynamic boundary layer thickness, δ.

Local skin friction coefficient:

Remembering that local skin friction coefficient is defined as:

$$C_{fx} = \frac{\tau_w}{\frac{1}{2} \cdot \rho \cdot U^2}$$

and using Eq. 9.72 for τ_w and Eq. 9.73 for δ, we get:

$$C_{fx} = 0.0576\, Re_x^{\frac{-1}{5}} \quad ...(9.74)$$

Average value of skin friction coefficient:

Average value of C_{fx} over length L is given by:

$$C_{fa} = \frac{1}{L} \cdot \int_0^L C_{fx}\, dx$$

Substituting for C_{fx} from Eq. 9.74 and performing the integration, we get:

$$C_{fa} = 0.072 \cdot Re_L^{\frac{-1}{5}} \qquad (for\ 5 \times 10^5 < Re_L < 10^7 \ldots(9.75))$$

Eq. 9.75 is valid for $5 \times 10^5 < Re_L < 10^7$ and $0.6 < Pr < 60$.

For values of Re_L between 10^7 and 10^9 following equation is suggested by Prandtl and Schlichting:

$$C_{fa} = \frac{0.455}{(\log(Re_L))^{2.58}} \qquad (for\ 10^7 < Re_L < 10^9 \ldots(9.76))$$

where $\log(Re_L)$ is the logarithm to base 10.

Local and average Nusselt numbers:

Local Nusselt number is calculated by applying Colburn analogy (which we shall study in the next section). We get:

$$Nu_x = \frac{h_x \cdot x}{k} = 0.0288 \cdot Re_x^{0.8} \cdot Pr^{\frac{1}{3}} \qquad (0.6 < Pr < 60 \ldots(9.77))$$

and,

$$Nu_{avg} = \frac{h_a \cdot L}{k} = 0.036 \cdot Re_L^{0.8} \cdot Pr^{\frac{1}{3}} \qquad \ldots(9.78)$$

For Eqs. 9.77 and 9.88, remember: $5 \times 10^5 < Re_L < 10^7$ and $0.6 < Pr < 60$

Local and average heat transfer coefficients:

These are determined from:

$$h_x = 0.0288 \cdot \left(\frac{k}{x}\right) \cdot Re_x^{0.8} \cdot Pr^{\frac{1}{3}} \qquad \ldots(9.79)$$

and,

$$h_a = 0.036 \cdot \left(\frac{k}{L}\right) \cdot Re_L^{0.8} \cdot Pr^{\frac{1}{3}} \qquad \ldots(9.80)$$

For an unheated starting length of x_0:

In turbulent flow, when heating starts from an initial length of x_0, i.e. thermal boundary layer begins at $x = x_0$:

$$Nu_x = \frac{0.0288 \cdot Re_x^{0.8} \cdot Pr^{\frac{1}{3}}}{\left[1 - \left(\frac{x_0}{x}\right)^{\frac{9}{10}}\right]^{\frac{1}{9}}} \qquad \ldots(9.81)$$

Note: both x_0 and x are measured from the leading edge of the plate.

Some comments on the variation of local heat transfer coefficient and local friction coefficient along the length x from the leading edgė of the plate, in laminar and turbulent flow are appropriate:

(a) In laminar flow, we have:

$$C_{fx} = \frac{\tau}{\left(\frac{\rho \cdot U^2}{2}\right)} = \frac{0.664}{\sqrt{Re_x}} \qquad \ldots(9.31)$$

and

$$\frac{h \cdot x}{k} = Nu_x = 0.332 \cdot \sqrt{Re_x} \cdot Pr^{0.333} \qquad \ldots(9.38)$$

i.e. in laminar flow, local friction coefficient varies as $x^{-1/2}$; likewise, from Eq. 9.38, it is clear that local heat transfer coefficient also varies as $x^{-1/2}$. Of course, at the leading edge (i.e. at $x = 0$), both these values are infinite and then decrease along the length of the plate according to $x^{-1/2}$.

(b) In turbulent flow, we have:

$$C_{fx} = 0.0576 \cdot Re_x^{\frac{-1}{5}} \quad ...(9.74)$$

and,

$$h_x = 0.0288 \cdot \left(\frac{k}{x}\right) \cdot Re_x^{0.8} \cdot Pr^{\frac{1}{3}} \quad ...(9.79)$$

i.e. in turbulent flow, both the local friction coefficient and the local heat transfer coefficient vary as $x^{-0.2}$. So, as we proceed along the length of the plate, initially, starting from the leading edge, the flow is laminar where both the local friction and heat transfer coefficients vary as $x^{-1/2}$; then, the flow turns turbulent when the critical distance is reached, and both the local friction and heat transfer coefficients reach their highest values at this point and then they decrease along the distance according to: $x^{-0.2}$. This is shown graphically in Fig. 9.13. In Fig. 9.13, the transition region is also shown.

For uniform heat flux conditions:

Local Nusselt number increases by about 4% over the value for constant wall temperature, and is given by:

$$Nu_x = 0.0308 \cdot Re_x^{0.8} \cdot Pr^{\frac{1}{3}} \quad ...(9.81a)$$

Also, in the above equations, it is assumed that flow over the plate is turbulent over the entire plate from the leading edge itself, or alternatively, region of laminar flow is too small compared to the region of turbulent flow.

9.8.3.2 Combined laminar and turbulent flow over a flat plate. As explained earlier, for a flow over a flat plate, the flow at the leading edge starts as laminar and after a critical distance x_c the flow becomes turbulent. If the distance over which the flow is laminar is not negligible as compared to the distance over which the flow is turbulent (i.e. the plate is long enough to cause the boundary layer to become turbulent, but not long enough to neglect the length over which the flow is laminar), average friction coefficient and average Nusselt number over the entire plate are determined by integrating the respective local values over two regions, i.e. the laminar region, $0 < x < x_c$ and, the turbulent region, $x_c < x < L$, as shown below:

$$C_{fa} = \frac{1}{L} \cdot \left[\int_0^{x_c} C_{fx\,\text{laminar}}\, dx + \int_{x_c}^{L} C_{fx\,turb}\, dx\right] \quad ...(9.82)$$

and,

$$h = \frac{1}{L} \cdot \left[\int_0^{x_c} h_{x\,\text{laminar}}\, dx + \int_{x_c}^{L} h_{x\,turb}\, dx\right] \quad ...(9.83)$$

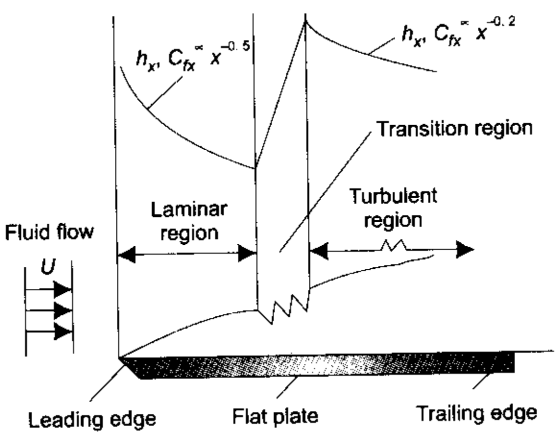

FIGURE 9.13 Variation of local friction and heat transfer coefficients for flow over a flat plate

If we perform the integration taking the value of critical Reynolds number, Re_c as 5×10^5, we get for the average friction coefficient and average Nusselt number, the following relations:

$$C_{fa} = \frac{0.074}{Re_L^{\frac{1}{5}}} - \frac{1742}{Re_L} \quad ...(5 \times 10^5 < Re_L < 10^7)...(9.84a)$$

Another, more general relationship used for critical Reynolds number other than 5×10^5 is:

$$C_{fa} = \frac{0.455}{(\log(Re_L))^{2.584}} - \frac{A}{Re_L} \cdot 8.28 \quad ...(9.84b)$$

where value of A is 1050, 1700, 3300 and 8700 respectively for values of Re_c equal to 3×10^5, 5×10^5, 1×10^6, and 3×10^6.

and, for critical Reynolds number of 5×10^5, average Nusselt number over the entire plate is

$$Nu_{avg} = \frac{h \cdot L}{k} = \left(0.036 \cdot Re_L^{\frac{4}{5}} - 836\right) \cdot Pr^{\frac{1}{3}} \quad ...(0.6 < Pr < 60), \textit{ and } (5 \times 10^5 < Re_L < 10^7)...(9.85a)$$

and, more generally, for critical Reynolds numbers other than 5×10^5:

$$Nu_{avg} = Pr^{\frac{1}{3}} \cdot \left(0.036 \cdot Re_L^{0.8} - A\right) \quad ...(9.85b)$$

where $A = 0.036 \cdot Re_c^{0.8} - 0.664 \cdot Re_c^{0.5}$

Example 9.8. A refrigerated truck is moving at a speed of 85 km/h where ambient temperature is 50°C. The body of the truck is of rectangular shape of size 10 m (L) × 4 m(W) × 3 m(H). Assume the boundary layer is turbulent and the wall surface temperature is at 10°C. Neglect heat transfer from vertical front and backside of truck and flow of air is parallel to 10 m long side. Calculate heat loss from the four surfaces.

For turbulent flow over flat surfaces: $Nu = 0.036.Re^{0.8}.Pr^{0.33}$

Average properties of air at 30°C: $\rho = 1.165$ kg/m^3, $C_p = 1.005$ kJ/kgK, $\nu = 16.10^{-6}$ m^2/s, $Pr = 0.701$

(M.U. Dec. 1999).

Solution.

Data:

$T_s := 10°C \quad T_a := 50°C \quad L := 10$ m $\quad W := 4$ m $\quad H := 3$ m $\quad A := L \cdot (W + H) \cdot 2$ m^2 i.e. $A = 140$ m^2

$$T_f := \frac{T_s + T_a}{2} + 273 \quad T_f = 303 \text{ K} \quad \textit{(film temperature)}$$

Truck is moving at a speed of 85 km/h, i.e.

$$U := \frac{85000}{3600} \quad \text{i.e.} \quad U = 23.611 \text{ m/s} \quad \textit{(velocity of air over the surfaces)}$$

Properties at T_f by data:

$\nu := 16 \times 10^{-6}$ m^2/s $\quad \rho := 1.165$ kg/m^3 $\quad C_p := 100\ 5$ J/kgK $\quad Pr := 0.701$

$$k := \frac{\nu \cdot \rho \cdot C_p}{Pr} \quad k = 0.02672 \text{ W/mK} \quad \textit{(thermal conductivity of air)}$$

Check if flow is laminar or turbulent:

Reynolds number:

$$Re_L := \frac{L \cdot U}{\nu}$$

i.e. $\quad Re_L = 1.476 \times 10^7 ...> 5 \times 10^5$

i.e. flow is turbulent since Reynolds number is more than 5×10^5

Heat transfer:

For turbulent flow, we have:

$$Nu := 0.036 \cdot Re_L^{0.8} \cdot Pr^{\frac{1}{3}} \quad \textit{(Nusselts number)}$$

i.e. $\quad Nu = 1.738 \times 10^4$

$$h := \frac{k \cdot Nu}{L}\ \text{W/(m}^2\text{K)}$$ *(heat transfer coefficient)*

i.e. $h = 46.448\ \text{W/(m}^2\text{K)}$ *(heat transfer coefficient)*

Therefore,

$$Q := h \cdot A \cdot (T_a - T_s)\ \text{W}$$ *(total heat transfer rate from all the four surfaces)*

i.e. $Q = 2.60 \times 10^5$ **W** ***(total hat transfer rate from all the four surfaces.)***

Also, find out the power required to overcome wind resistance:

We have, average skin friction coefficient given by:

$$C_{fa} := \frac{0.455}{(\log(Re_L))^{2.58}}$$ *(for $10^7 < Re_L < 10^9$...(9.76))*

i.e. $C_{fa} = 2.82 \times 10^{-3}$ *(average skin friction coefficient)*

Therefore, Drag force: $F_D := C_{fa} \cdot \frac{\rho \cdot A \cdot U^2}{2}$ *(N)*

i.e. $F_D = 128.406$ N *(Drag force)*

Therefore, Power: $P := F_D \cdot U$, W

i.e. $P = 3.032 \times 10^3$ W *(Power required to overcome air resistance)*

i.e. $P = 3.032$ **kW** ***(Power required to overcome air resistance.)***

Example 9.9. A flat plate, 1 m wide and 1.5 m long is maintained at 90°C in air with free stream temperature of 10°C, flowing along 1.5 m side of the plate. Determine the velocity of air required to have a rate of energy dissipation as 3.75 kW. Use correlations:

$Nu_L = 0.664\ Re^{0.5}\ Pr^{1/3}$ for Laminar flow, and

$Nu_L = [0.036\ Re^{0.8} - 836].Pr^{1/3}$ for turbulent flow.

Take average properties of air at 50°C: $\rho = 1.0877$ kg/m³, $C_p = 1.007$ kJ/kgK,

$\mu = 2.029.10^{-5}$ kg/m.s, $Pr = 0.703$, $k = 0.028$ W/mK [P.U.; 1995]

Solution.

Data:

$T_s := 90°C$ $T_a := 10°C$ $L := 1.5$ m $W := 1$ m $Q := 3750$ W $A := L \cdot (W) \cdot 2\ \text{m}^2$ i.e. $A = 3\ \text{m}^2$

$$T_f := \frac{T_s + T_a}{2} + 273 \quad T_f = 323\ \text{K}$$ *(film temperature)*

Properties at T_f: by data:

$\mu := 2.029 \times 10^{-5}$ kg/(ms) $\rho := 1.0877$ kg/m³ $C_p := 1007$ J/kgK $Pr := 0.703$ $k := 0.028$ W/mK

Nusselt number:

We have, for convection heat transfer:

$$Q = h_a \cdot A \cdot (T_s - T_a)\ \text{W}$$ *(from Newton's Law of Cooling)*

i.e. $h_a := \frac{Q}{A \cdot (T_s - T_a)}$ W/(m²C) *(average heat transfer coefficient)*

i.e. $h_a = 15.625$ W/(m²C) *(average heat transfer coefficient)*

Therefore, **Nusselt number:**

$$Nu_L := \frac{h_a \cdot L}{k}$$

i.e. $Nu_L = 837.054$

Now, we do not know if the flow is laminar or turbulent. To determine this, we need the Reynolds number. But, we do not know the velocity to determine the Reynolds number. So, we shall first assume the flow to be laminar and then check if the Reynolds number works out to be less than the critical Reynolds number (i.e. 5×10^5):

For Laminar flow:

$$Nu_L = 0.664 \cdot Re_L^{0.5} \cdot Pr^{\frac{1}{3}}$$

i.e.

$$Re_L := \left[\frac{Nu_L}{0.664 \cdot Pr^{\frac{1}{3}}}\right]^2$$

i.e. $Re_L = 2.01 \times 10^6$

This value of Reynolds number is greater than the critical Reynolds number of 5×10^5. Therefore, the assumption that the flow is laminar is wrong.

Then, for turbulent flow, we use the relation:

$$Nu_L := \left(0.036 \cdot Re_L^{0.8} - 836\right) \cdot Pr^{\frac{1}{3}}$$

Therefore,

$$Re_L := \left[\frac{\left[\frac{Nu_L}{Pr^{\frac{1}{3}}} + 836 \right]}{0.036} \right]^{\frac{1}{0.8}}$$

i.e. $Re_L = 7.36 \times 10^5 > 5 \times 10^5$ *(Therefore, assumption of turbulent flow is correct.)*

To find the velocity of air:

We have: $Re_L = \dfrac{\rho \cdot U \cdot L}{\mu}$ *(Reynolds number, by definition)*

i.e. $U := \dfrac{Re_L \cdot \mu}{\rho \cdot L}$ m/s *(velocity of air)*

i.e. $U = 9.152$ **m/s** ***(velocity of air.)***

Example 9.10. Air at 30°C flows over a flat plate, 0.4 m wide and 0.75 m long with a velocity of 20 m/s. Determine the heat flow rate from the surface of the plate assuming that the flow is parallel to the 0.75 m side. Plate is maintained at 90°C. Use correlations:

$Nu_L = 0.664\ Re^{0.5}\ Pr^{1/3}$ for Laminar flow, and

$Nu_L = [0.036\ Re^{0.8} - 836].Pr^{1/3}$ for turbulent flow.

Take average properties of air at 60°C: $\rho = 1.06$ kg/m^3, $C_p = 1.008$ kJ/kgK,

$\nu = 18.97 \cdot 10^{-6}$ m^2/s, $Pr = 0.708$, $k = 0.0285$ W/mK [M.U.]

Solution.

Data:

$T_s := 90°C$ $T_a := 30°C$ $U := 20$ m/s $L := 0.75$ m $W := 0.4$ m

$$T_f := \frac{T_s + T_a}{2} + 273 \quad T_f = 333 \text{ K}$$ *(film temperature)*

Properties at Tf by data:

$\nu := 18.97 \times 10^{-6}$ m^2/s $\rho := 1.06$ kg/m^3 $C_p := 1008$ J/kgK $Pr := 0.708$ $k := 0.0285$ W/mK

First, let us find out the distance from the leading edge at which the flow turns turbulent, assuming the critical Reynolds number to be 5×10^5, i.e. Lc at which the critical Reynolds number is reached:

$Re_c := 5 \times 10^5$ *(critical Reynolds number)*

$$Re_c = \frac{L_c \cdot U}{\nu}$$

i.e. $L_c := \dfrac{Re_c \cdot \nu}{U}$

i.e. $L_c = 0.474$ m *...length from leading edge, at which flow turns turbulent.*

i.e. along the length of the plate, for a distance of 0.474 m, the flow is laminar. This distance can not be neglected as compared to the total length of the plate of 0.75 m. Therefore, combined effect of laminar and turbulent boundary layer flow has to be considered.

For the case of combined laminar and turbulent boundary layers, we have:

$Re_L := \dfrac{L \cdot U}{\nu}$ i.e. $Re_L = 7.907 \times 10^5$ *(Reynolds number at the end of plate)*

$$Nu_{avg} = \frac{h_a \cdot L}{k} = \left(0.036 \cdot Re_L^{\frac{4}{5}} - 836\right) \cdot Pr^{\frac{1}{3}}$$ ...(0.6 < *Pr* < 60), *and* (5 × 10^5 < Re_L < 10^7) ...(9.85a)

Therefore, $Nu_{avg} := \left(0.036 \cdot Re_L^{\frac{4}{5}} - 836\right) \cdot Pr^{\frac{1}{3}}$ *(average Nusselt number over the entire plate)*

i.e. $Nu_{avg} = 932.666$

or, $h_a := \frac{Nu_{avg} \cdot k}{L}$ W/(m²K) *(average heat transfer coefficient over the entire plate)*

i.e. $h_a = 35.441$ W/(m²K) *(average heat transfer coefficient over the entire plate)*

Therefore, **heat transfer rate:**

$Q := h_a (L \cdot W) \cdot (T_s - T_a)$ W *(heat transfer rate from the entire plate)*

i.e. $Q = 637.944$ **W** ***(heat transfer rate from the entire plate)***

Alternatively, we can calculate the heat transferred by the laminar and turbulent regions separately, and then add them up, to get the total heat transfer rate for the whole plate:

For laminar flow region (i.e. upto a distance of 0.474 m along the length):

$Nu_{lam} := 0.664 \cdot Re_c^{0.5} \cdot Pr^{\frac{1}{3}}$ *(Nusselts number for Laminar region)*

i.e. $Nu_{lam} = 418.47$

Therefore, $h_{a\_lam} := \frac{Nu_{lam} \cdot k}{L_c}$ W/(m²K) *(average heat transfer coefficient over the laminar region)*

i.e. $h_{a\_lam} = 25.148$ W/(m²k) *(average heat transfer coefficient over the laminar region)*

Therefore, heat transfer rate for the laminar region, Q_1:

$Q_1 := h_{a\_lam} \cdot (W \cdot L_c)(T_s - T_a)$ W *(heat transfer from laminar region)*

i.e. $Q_1 = 286.233$ W *(heat transfer from laminar region)*

For turbulentr flow region (i.e. from a distance of 0.474 m upto the end of plate):

Local Nusselt number for the turbulent region is given by:

$$Nu_x = \frac{h_x \cdot x}{k} = 0.0288 \cdot Re_x^{0.8} \cdot Pr^{\frac{1}{3}}$$ ...(9.77)

i.e.
$$h_x = 0.0288 \cdot \left(\frac{k}{x}\right) \cdot \left(\frac{U \cdot x}{\nu}\right)^{0.8} \cdot Pr^{\frac{1}{3}}$$

i.e.
$$h_x = 0.0288 \cdot k \cdot Pr^{\frac{1}{3}} \cdot \left(\frac{U}{\nu}\right)^{0.8} \cdot x^{-0.2}$$

Therefore, average value of heat transfer coefficient for turbulent region is obtained as

$$h_{a\_turb} = 0.0288 \cdot k \cdot Pr^{\frac{1}{3}} \cdot \left(\frac{U}{\nu}\right)^{0.8} \cdot \frac{1}{(L - L_c)} \cdot \int_{L_c}^{L} x^{-0.2}\, dx$$

i.e.
$$h_{a\_turb} = 0.0288 \cdot k \cdot Pr^{\frac{1}{3}} \cdot \left(\frac{U}{\nu}\right)^{0.8} \cdot \frac{1}{(L - L_c)} \cdot \frac{(L^{0.8} - L_c^{0.8})}{0.8}$$

i.e.
$$h_{a\_turb} = 0.036 \cdot k \cdot Pr^{\frac{1}{3}} \cdot \frac{1}{(L - L_c)} \cdot \left[\left(\frac{U \cdot L}{\nu}\right)^{0.8} - \left(\frac{(U \cdot L_c)}{\nu}\right)^{0.8}\right]$$

i.e.
$$h_{a\_turb} = 0.036 \cdot k \cdot Pr^{\frac{1}{3}} \cdot \frac{1}{(L - L_c)} \cdot [(Re_L)^{0.8} - (Re_c)^{0.8}]$$

i.e.
$$\frac{h_{a\_turb} \cdot (L - L_c)}{k} = 0.036 \cdot [(Re_L)^{0.8} - (Re_c)^{0.8}] \cdot Pr^{\frac{1}{3}}$$

Note that $\frac{h_{a\_turb} \cdot (L - L_c)}{k}$ is the average Nusselts number for turbulent region

Heat transfer rate for the turbulent region Q_2

$$h_{a\_turb} := 0.036 \cdot k \cdot Pr^{\frac{1}{3}} \cdot \frac{1}{(L - L_c)} \cdot [(Re_L)^{0.8} - (Re_c)^{0.8}]$$

i.e. $h_{a\_turb} = 53.229 \text{ W/(m}^2\text{K)}$ *(average heat transfer coefficient over the turb. region)*

$Q_2 := h_{a\_turb} \cdot (W) \cdot (L - L_c) \cdot (T_s - T_a)$ W *(heat transfer rate for turbulent region)*

i.e. $Q_2 = 352.269$ W *(heat transfer rate for turbulent region)*

Therefour, total heat transfer rate, Q:

$Q := Q_1 + Q_2$

i.e. $Q = 638.502$ **W** ***(total heat transfer rate for the plate)***

This value matches with the value obtained earlier by direct formula.

To show graphically the variation of local heat transfer coefficient over the entire length of plate:

We have stated earlier that the local heat transfer coefficient for the laminar region varies as $x^{-0.5}$ and that for the turbulent region varies as $x^{-0.2}$. Let us illustrate this graphically, using Mathcad.

For laminar region, i.e. from $x_1 = 0$ to $x_1 = 0.474$ m along the length of plate, local heat transfer coefficient as a function of x is written as:

$$h_{x\_lam}(x_1) := 0.332 \cdot \frac{k}{x_1} \cdot \left(\frac{x_1 \cdot U}{\nu}\right)^{0.5} \cdot Pr^{\frac{1}{3}}$$

For turbulent region, i.e. from $x_2 = 0.474$ m to $x_2 = 0.75$ m along the length of plate, local heat transfer coefficient as a function of x is written as:

$$h_{x\_turb}(x_2) := 0.088 \cdot \frac{k}{x_2} \cdot \left(\frac{x_2 \cdot U}{\nu}\right)^{0.8} \cdot Pr^{\frac{1}{3}}$$

Now, for the first case, let us define a range variable x_1 varying from $x_1 = 0$ to $x_1 = 0.474$ m and draw the graph by choosing the x–y graph from the graph pallete, and filling up the place holder on the x-axis with x_1 and the place holder on the y-axis with $h_{x\text{-lam}}(x_1)$; then for the second case, again define a range variable x_2 varying from $x_2 = 0.474$ m to $x_2 = 0.75$ m and in the place holder on the x-axis, put a comma after x_1 and type x_2 and in the place holder on the y-axis put a comma after $h_{x\text{-lam}}(x_1)$ and then type $h_{x\text{-turb}}(x_2)$. Click anywhere outside the graph region and immediately the graphs appear. See Fig. Ex. 9.10

$x_1 := 0, 0.01, ..., 0.47$ *(define range variable x_1 varying from 0 to 0.47 m, with an increment of 0.01 m)*

$x_2 := 0.47, 0.48, ..., 0.75$ *(define range variable x_2 varying from 0.47 to 0.75 m, with an increment of 0.01 m)*

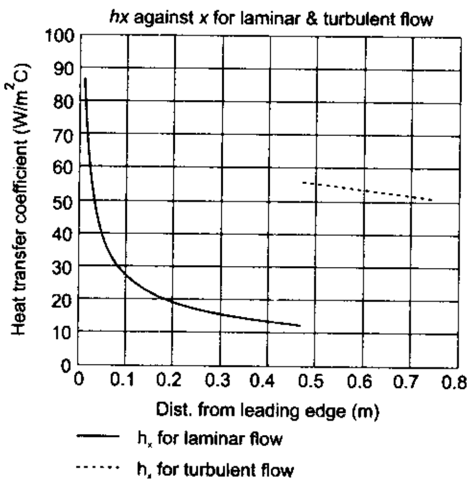

FIGURE Example 9.10 Variation of local heat transfer coefficient along the length of a flat plate for laminar and turbulent boundary layer heat transfer

Note:

(i) In the above Fig. Example 9.10 first portion of the curve is for laminar boundary layer heat transfer and the second portion is for turbulent boundary layer heat transfer. In laminar portion, the local heat transfer falls steeply, as $x^{-0.5}$, along the length upto the critical distance; once the critical distance is reached, the boundary layer turns turbulent and the local heat transfer coefficient suddenly increases to a high value and then, with increasing x the local heat transfer coefficient drops more gradually (as $x^{-0.2}$) as compared to the laminar portion.

(ii) In the laminar region, the heat transfer coefficient varies from an infinite value at $x = 0$ to about 12 W/(m²C) at $x = 0.47$ m. And, average heat transfer coefficient for the laminar region, as already calculated, is 25.148 W/(m²C).

(iii) In the turbulent region, the heat transfer coefficient varies from a value of about 56 W/(m²C) at $x = 0.47$ m to about 51 W/(m²C) at $x = 0.75$ m. And, average heat transfer coefficient for the turbulent region, as already calculated, is 53.229 W/(m²C).

(iv) Average heat transfer coefficient over the entire plate, for the combined laminar and turbulent regions, is 35.441 W/(m²C).

9.8.4 Analogy Between Momentum and Heat Transfer

We have shown that the two-dimensional equations for the momentum transport and energy transport have identical forms. It is reasonable to assume that their solutions also must have some correspondence to each other.

Solution of momentum equation leads us to a relation for the skin friction coefficient and the drag force; similarly, solution of the energy equation leads us to an expression for the heat transfer coefficient. So, we seek an analogy or relation between the fluid friction and heat transfer coefficients:

9.8.4.1 Relation between the fluid friction and heat transfer coefficient in laminar flow for a flat plate. Recollect that the average Nusselt number for laminar flow over a flat plate is given by:

$$Nu_a = 0.664 \cdot \sqrt{Re_L} \cdot Pr^{0.333} \quad ...(9.41)$$

This can be rewritten as:

$$\frac{Nu_a}{Re_L \cdot Pr} = 0.664 \cdot Re_L^{\frac{-1}{2}} \cdot Pr^{\frac{-2}{3}} \quad ...(a)$$

Now, the LHS of Eq. a is a dimensionless number known as "Stanton number", St_a.

Substituting for Nu_a, Re_L and Pr from their respective definitions, we get:

$$St_a = \frac{h_a}{\rho \cdot U \cdot C_p} = \frac{h_a}{G \cdot C_p} \quad ...(b)$$

where $G = \rho.U$ (kg/(m²s), is known as mass velocity.

Therefore, we write:

$$St_a = 0.664 \cdot Re_L^{\frac{-1}{2}} \cdot Pr^{\frac{-2}{3}} \quad ...(c)$$

i.e.

$$St_a \cdot Pr^{\frac{2}{3}} = 0.664 \cdot Re_L^{\frac{-1}{2}} \quad ...(d)$$

However, we have already shown that:

$$C_{fa} = \frac{1}{L} \cdot \int_0^L C_{fx}\, dx = \frac{1.328}{\sqrt{Re_L}} \quad (9.32)$$

i.e.

$$\frac{C_{fa}}{2} = 0.664 \cdot Re_L^{\frac{-1}{2}} \quad ...(e)$$

Then, comparing Eqs. d and e, we can write:

$$St_a \cdot Pr^{\frac{2}{3}} = \frac{C_{fa}}{2} \quad ...(9.86)$$

This relation is known as **'Colburn analogy'** and it gives a simple relation between the heat transfer coefficient and the friction coefficient. Eq. 9.86 is valid for values of Pr between 0.6 and 50. LHS of Eq. 9.86 is also known as 'Colburn j-factor', and is generally used to correlate heat transfer coefficient with Reynolds number.

Note the important significance of this analogy: just by knowing the friction coefficient, one can predict the heat transfer coefficient for that situation; and conducting experiments to determine friction coefficient is, many times, practically much easier than conducting experiments to determine heat transfer coefficients.

9.8.4.2 Reynolds and Colburn analogies for turbulent flow over a flat plate. Considering the laminar sub-layer adjacent to the plate surface, we have the relation for shear stress, along the X-direction and at $y = 0$:

$$\tau_x = \mu \cdot \frac{du}{dy} \quad \text{...(a)}$$

and, heat flux at the surface in the y-direction is:

$$q = -k \cdot \frac{dT}{dy} \quad \text{...(b)}$$

Combining Eqs. a and b:

$$q = -\tau \cdot \frac{k}{\mu} \cdot \frac{dT}{du} \quad \text{...(c)}$$

Now, if Prandtl number is unity, i.e. if $C_p = k/\mu$, we replace (k/μ) in Eq. c by C_p, and separating the variables, we write, assuming q and τ to be constant:

$$\frac{q_s}{\tau_s \cdot C_p} \cdot du = -dT \quad \text{...(d)}$$

In Eq. d, subscript s indicates that q and τ are considered at the surface of the plate.

Integrating Eq. d between the limits $u = 0$ when $T = T_s$ and $u = U$ when $T = T_a$, gives:

$$\frac{q_s}{\tau_s \cdot C_p} \cdot U = (T_s - T_a) \quad \text{...(e)}$$

However, by definition, the local heat transfer and friction coefficients are given by:

$$h_x = \frac{q_s}{T_s - T_a} \quad \text{and,} \quad \tau_x = C_{fx} \cdot \frac{\rho \cdot U^2}{2}$$

Then, Eq. e can be written as: $h_x \cdot U = C_{fx} \cdot \rho \cdot \frac{U^2}{2} \cdot C_p$

i.e.

$$St_x = \frac{h_x}{\rho \cdot U \cdot C_p} = \frac{Nu_x}{Re_x \cdot Pr} = \frac{C_{fx}}{2} \quad \text{...(9.87)}$$

Eq. 9.87 is known as **'Reynold's analogy'** and it gives a relation between Nusselts number (i.e. heat transfer coefficient) and the friction coefficient. Note that Reynolds analogy was derived with the assumption that $Pr = 1$ and is valid for most of the gases.

However, when the Prandtl number is different from unity, **Colburn's analogy,** i.e.

$$St_x \cdot Pr^{\frac{2}{3}} = \frac{C_{fx}}{2} \quad \text{...(9.88)}$$

is applied. This is valid for values of Pr between 0.6 and 50.

In practice, to apply the analogy between momentum and heat transfer, it is necessary to know the friction coefficient C_{fx}. For turbulent flow over a flat plate, we have the empirical relation for local friction coefficient:

$$C_{fx} = 0.0576 \cdot Re_x^{\frac{-1}{5}} \quad \text{...(9.74)}$$

Eq. 9.74 is valid for: $5 \times 10^5 < Re_x < 10^7$.

Example 9.11. Air at 27°C and 1 atm flows over a flat plate at a speed of 2 m/s. Assuming that the plate is heated over its entire length to a temperature of 60°C, calculate the heat transfer for the first 0.4 m of the plate. Also, compute the drag force exerted on the first 0.4 m of the plate using Reynolds analogy. Assume air to be a perfect gas with $R = 287$ J/kgK and $C_p = 1.006$ kJ/kgK. (M.U. May 1999).

Solution.

Data:

$T_s := 60°C \quad T_a := 27°C \quad U := 2 \text{ m/s} \quad L := 0.4 \text{ m} \quad B := 1 \text{ m}$

$$T_f := \frac{T_s + T_a}{2} + 273 \quad T_f = 316.5 \text{ K}$$ *(film temperature)*

$$R := 287 \text{ J/(kgK)}$$ *(Gas constant for air)*

$$P := 1.01325 \times 10^5 \text{ Pa}$$ *(atmospheric pressure)*

Properties at T_f:

$\nu := 17.2 \times 10^{-6} \text{ m}^2/\text{s} \quad C_p := 1006 \text{ J/kgK} \quad Pr := 0.71 \quad k := 0.0271 \text{ W/mK} \quad \rho := \frac{P}{R \cdot T_f} \text{ kg/m}^3$

i.e. $\rho := 1.115 \text{ kg/m}^3$

Reynolds number:

$$Re_L := \frac{L \cdot U}{\nu}$$

i.e. $Re_L = 4.65 \times 10^4$ *(less than $Re_{cr} = 5 \times 10^5$)*

i.e. flow is laminar.

Therefore, we have for average Nusselts number:

$$Nu_a := 0.664 \cdot \sqrt{Re_L} \cdot Pr^{\frac{1}{3}} \quad \text{...(9.41)}$$

i.e. $Nu_a = 127.752$

Average heat transfer coefficient:

$$h_a := \frac{Nu_a \cdot k}{L} \text{ W/(m}^2\text{K)}$$ *(average heat transfer coefficient)*

i.e. $h_a = 8.655 \text{ W/(m}^2\text{K)}$ *(average heat transfer coefficient)*

Heat transfer rate:

$$Q := h_a \cdot (L \cdot B) \cdot (T_s - T_a) \text{ W}$$ *(heat transfer rate for the first 0.4 m length)*

i.e. $Q = 114.249$ **W** ***(heat transfer rate for the first 0.4 m length.)***

To calculate the drag force:

We have, for mass velocity:

$$G := \rho \cdot U \text{ kg/(sm}^2\text{)}$$ *(mass velocity)*

Therefore, Stanton number, by definition:

$$St := \frac{h_a}{G \cdot C_p}$$

i.e. $St = 3.856 \times 10^{-3}$ *(Stanton number)*

Check: $St = \frac{Nu_a}{Re_L \cdot Pr} = 3.869 \times 10^{-3}$ *(checks.)*

By Reynolds Analogy: $St = C_f/2$

i.e. $C_f := 2 \cdot St$

i.e. $C_f = 7.713 \times 10^{-3}$ *(skin friction coefficient)*

Drag force:

$$F_D := C_f \cdot \rho \cdot \frac{U^2}{2} \cdot (L \cdot B)$$

i.e. $F_D = 6.883 \times 10^{-3}$ **N** ***(drag force exerted on first 0.4 m length.)***

9.9 Flow Across Cylinders, Spheres and Other Bluff Shapes and Packed Beds

So far, we studied external flow over a flat plate. Next, we shall consider flow across cylinders, spheres and other bluff shapes such as disk or half cylinder. These cases are of considerable practical importance. Case of single cylinder in cross flow is identical to the case of cooling of an electrical cable by forced convection by air flowing across it; also determination of local velocities in a flow by 'hot wire anemometer' involves the heat transfer from a single platinum wire maintained at a constant temperature (or by passing a constant current through it) and correlating the change in current (or change in resistance) to the velocity of flow. Heat transfer from a sphere is important when we are interested in performance of systems where clouds of particles are heated or cooled in a stream of fluid. Such an understanding is generally required when we correlate data for heat transfer in fluid beds, especially in the field of chemical engineering. If the particle is of an irregular shape, then an equivalent diameter is used in place of sphere diameter, i.e. D is taken as the diameter of an equivalent sphere that has the same surface area as that of the irregular shape. Front portion of an aeroplane wing can be approximated as a half cylinder while calculating the local heat transfer coefficients over the forward portion of the wing.

9.9.1 Flow Across Cylinders and Spheres

Now, the characteristic length taken to calculate the Reynolds number is the external diameter D of the cylinder or sphere. And the Reynolds number is defined, as usual:

$$Re_D = \frac{U \cdot D}{\nu}$$

where U is the uniform velocity of flow as it approaches the cylinder or sphere.

The critical Reynolds number for flow across cylinder or sphere is:

$$Re_{cr} = 2 \times 10^5$$

i.e. upto $Re = 2 \times 10^5$, the boundary layer remains laminar and beyond this value, the boundary layer becomes turbulent.

Flow patterns for a flow across a cylinder are shown in Fig. 9.14. Fluid particles at the mid-plane of a stream approaching the cylinder strike the cylinder at the 'stagnation point' and come to a halt, thus increasing the pressure. Rest of the fluid branches around the cylinder forming a boundary layer that embraces the cylinder walls. Pressure decreases in the flow direction and the velocity increases. At very low free stream velocities ($Re < 4$), the fluid completely wraps around the cylinder; as the velocity increases, boundary layer detaches from the surface at the rear, forming a wake behind the cylinder. This point is called 'separation point'. Flow separation occurs at about $\theta = 80$ deg. when the boundary layer is laminar and at about $\theta = 140$ deg. when the boundary layer is turbulent.

Drag coefficient (C_D): Drag force for a cylinder in cross flow is primarily due to two effects: one, 'friction drag' due to the shear stress at the surface, and the other, 'pressure drag' due to the pressure difference between the stagnation point and the wake. At low Reynolds numbers (< 4), friction drag is predominant, and at high Reynolds numbers (> 5000), pressure drag is predominant. At the intermediate values of Re, both the effects contribute to the drag.

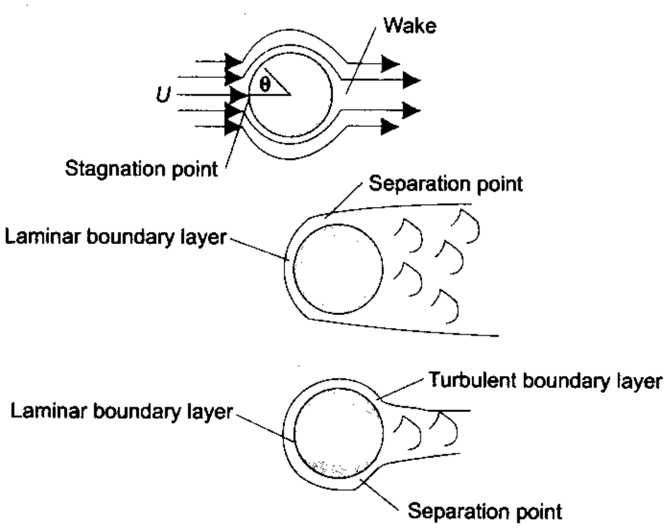

FIGURE 9.14 Flow patterns for cross flow over a cylinder

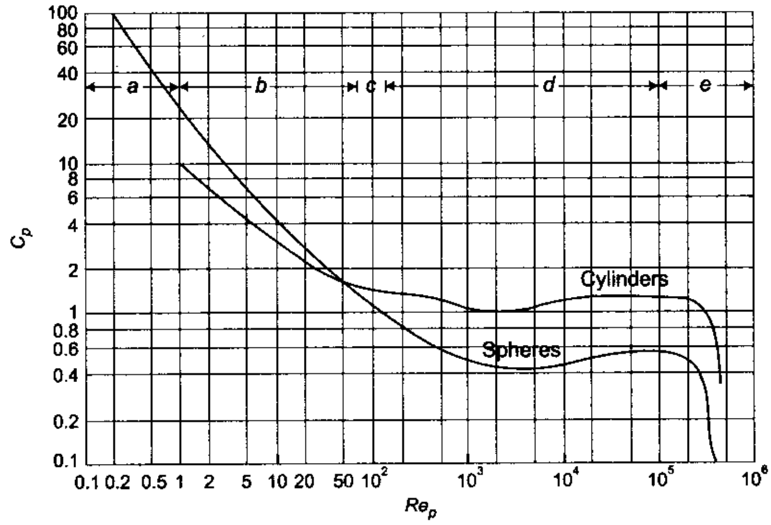

FIGURE 9.15 Drag Coefficient Versus Reynolds Number for Long Circular Cylinders and Spheres in Cross-Flow

Average drag coefficient C_D for cross flow over a cylinder and sphere are shown in Fig. 9.15. Then, the drag force acting on the body in cross flow is obtained from:

$$F_D = C_D \cdot A_N \cdot \frac{\rho \cdot U^2}{2}, \text{ N}$$

where A_N is the 'frontal area' i.e. area normal to the direction of flow.

$$A_N = L.D$$ ...*for a cylinder of length L*

and, $$A_N = \frac{\pi \cdot D^2}{4}$$ ...*for a sphere*

In Fig. 9.15, there are 5 sections, a, b, c, d and e shown. Comments corresponding to these sections of the figure are given below:

(a) At $Re < 1$, inertia forces are negligible and the flow adheres to the surface and drag is only by viscous forces. Heat transfer is purely by conduction.
(b) At Re = about 10, inertia forces become appreciable; now, pressure drag is about half of the total drag.
(c) At Re of the order of 100, vortices separate and the pressure drag predominates.
(d) At Re values between about 1000 and 100,000, skin friction drag is negligible compared to the pressure drag. Point of separation is at about $\theta = 80$ deg. measured from the stagnation point.
(e) At $Re > 100{,}000$, flow in the boundary layer becomes turbulent and the separation point moves to the rear.

Heat transfer coefficient: Because of the complex nature of flow, most of the results are empirical relations derived from experiments.

Variation of local Nusselt number around the periphery of a cylinder in cross flow is given in Fig. 9.16. *Nu* is high to start with at the stagnation point, then decreases as θ increases due to the thickening of laminar boundary layer. For the two curves at the bottom, minimum is reached at about $\theta = 80$ deg., the separation point in laminar flow. For the rest of the curves, there is a sharp increase at about $\theta = 90$ deg. due to transition from laminar to turbulent flow; *Nu* reaches a second minimum at about $\theta = 140$ deg. due to flow separation in turbulent flow, and thereafter increases with θ, due to intense mixing in the turbulent wake region.

Between $\theta = 0$ and 80 deg. empirical equation for local heat transfer coefficient is:

$$Nu(\theta) = \frac{h_c(\theta) \cdot D}{k} = 1.14 \cdot \left(\frac{\rho \cdot U \cdot D}{\mu}\right)^{0.5} \cdot Pr^{0.4} \cdot \left[1 - \left(\frac{\theta}{90}\right)^3\right] \quad \text{...(9.89)}$$